Intermediate Algebra Textbook for Skyline College

Adapted for Skyline College
from text created by
Department of Mathematics
College of the Redwoods

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Changes to the Skyline College Edition:
Chapter 2.5
Chapter 4.1

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Chapter 1 Factoring

The ancient Babylonians left the earliest evidence of the use of quadratic equa-tions on clay tablets dating back to 1800 BC. They understood how the area of a square changes with the length of its side. For example, they knew it was possible to store nine times more bales of hay in a square loft if the side of the loft was tripled in length. However, they did not know how to calculate the length of the side of a square starting from a given area. The word “quadratic” comes from “quadratus,” the Latin word for “square.” In this chapter, we will learn how to solve certain quadratic equations by factoring polynomials.

1.1 The Greatest CommonFactor

We begin this section with definitions of *factors* and *divisors*. Because $24 = 2 \cdot 12$, both 2 and 12 are factors of 24. However, note that 2 is also a divisor of 24, because when you divide 24 by 2 you get 12, with a remainder of zero. Similarly, 12 is also a divisor of 24, because when you divide 24 by 12 you get 2, with a remainder of zero.

Factors and divisors. Suppose m and n are integers. Then m is a *divisor* (*factor*) of n if and only if there exists another integer k so that $n = m \cdot k$.

The words divisor and factor are equivalent. They have the same meaning.

You Try It!

List the positive divisors of 18.

EXAMPLE 1. List the positive divisors (factors) of 24.

Solution: First, list all possible ways that we can express 24 as a product of two positive integers:

$$24 = 1 \cdot 24 \quad \text{or} \quad 24 = 2 \cdot 12 \quad \text{or} \quad 24 = 3 \cdot 8 \quad \text{or} \quad 24 = 4 \cdot 6$$

Therefore, the positive divisors (factors) of 24 are 1, 2, 3, 4, 6, 8, and 24.

Answer: $1, 2, 3, 6, 9$, and 18

□

You Try It!

List the positive divisors that 40 and 60 have in common.

EXAMPLE 2. List the positive divisors (factors) that 36 and 48 have in common.

Solution: First, list all positive divisors (factors) of 36 and 48 separately, then box the divisors that are in common.

$$\text{Divisors of 36 are: } \boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}, \boxed{6}, 9, \boxed{12}, 18, 36$$
$$\text{Divisors of 48 are: } \boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}, \boxed{6}, 8, \boxed{12}, 16, 24, 48$$

Therefore, the common positive divisors (factors) of 36 and 48 are 1, 2, 3, 4, 6, and 12.

Answer: $1, 2, 4, 5, 10$, and 20

□

Greatest common divisor. The greatest common divisor (factor) of a and b is the largest positive number that divides evenly (no remainder) both a and b. The greatest common divisor of a and b is denoted by the symbolism $\text{GCD}(a, b)$. We will also use the abbreviation $\text{GCF}(a, b)$ to represents the greatest common factor of a and b.

Remember, greatest common divisor and greatest common factor have the same meaning. In Example 2, we listed the common positive divisors of 36 and 48. The largest of these common divisors was 12. Hence, the greatest common divisor (factor) of 36 and 48 is 12, written $\text{GCD}(36, 48) = 12$.

With smaller numbers, it is usually easy to identify the greatest common divisor (factor).

You Try It!

State the greatest common divisor of 36 and 60.

EXAMPLE 3. State the greatest common divisor (factor) of each of the following pairs of numbers: (a) 18 and 24, (b) 30 and 40, and (c) 16 and 24.

Solution: In each case, we must find the largest possible positive integer that divides evenly into both the given numbers.

a) The largest positive integer that divides evenly into both 18 and 24 is 6. Thus, $\text{GCD}(18, 24) = 6$.

b) The largest positive integer that divides evenly into both 30 and 40 is 10. Thus, $\text{GCD}(30, 40) = 10$.

c) The largest positive integer that divides evenly into both 16 and 24 is 8. Thus, $\text{GCD}(16, 24) = 8$.

Answer: 12

□

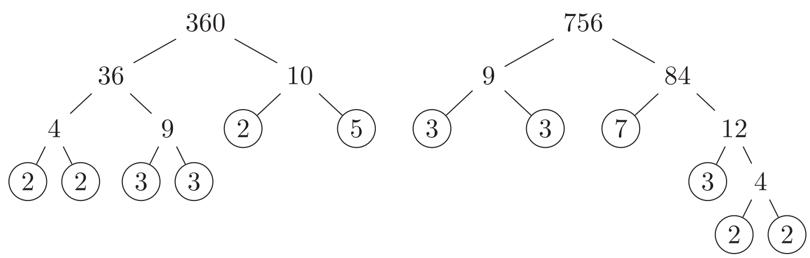
With larger numbers, it is harder to identify the greatest common divisor (factor). However, prime factorization will save the day!

You Try It!

Find the greatest common divisor of 120 and 450.

EXAMPLE 4. Find the greatest common divisor (factor) of 360 and 756.

Solution: Prime factor 360 and 756, writing your answer in exponential form.



Thus:

$$
\begin{aligned}
360 &= 2^3 \cdot 3^2 \cdot 5 \\
756 &= 2^2 \cdot 3^3 \cdot 7
\end{aligned}
$$

To find the greatest common divisor (factor), list each factor that appears in common to the *highest* power that appears in common.

In this case, the factors 2 and 3 appear in common, with 2^2 being the highest power of 2 and 3^2 being the highest power of 3 that appear in common. Therefore, the greatest common divisor of 360 and 756 is:

$$
\begin{aligned}
\text{GCD}(360, 756) &= 2^2 \cdot 3^2 \\
&= 4 \cdot 9 \\
&= 36
\end{aligned}
$$

Therefore, the greatest common divisor (factor) is $\text{GCD}(360, 756) = 36$. Note what happens when we write each of the given numbers as a product of the greatest common factor and a second factor:

$$
\begin{aligned}
360 &= 36 \cdot 10 \\
756 &= 36 \cdot 21
\end{aligned}
$$

In each case, note how the second second factors (10 and 21) contain no additional common factors.

Answer: 30

□

Finding the Greatest Common Factor of Monomials

Example 4 reveals the technique used to find the greatest common factor of two or more monomials.

Finding the GCF of two or more monomials. To find the greatest common factor of two or more monomials, proceed as follows:

1. Find the greatest common factor (divisor) of the coefficients of the given monomials. Use prime factorization if necessary.
2. List each variable that appears in common in the given monomials.
3. Raise each variable that appears in common to the highest power that appears in common among the given monomials.

You Try It!

EXAMPLE 5. Find the greatest common factor of $6x^3y^3$ and $9x^2y^5$.

Find the greatest common factor of $16xy^3$ and $12x^4y^2$.

Solution: To find the GCF of $6x^3y^3$ and $9x^2y^5$, we note that:

1. The greatest common factor (divisor) of 6 and 9 is 3.
2. The monomials $6x^3y^3$ and $9x^2y^5$ have the variables x and y in common.
3. The highest power of x in common is x^2. The highest power of y in common is y^3.

Thus, the greatest common factor is $\text{GCF}(6x^3y^3, 9x^2y^5) = 3x^2y^3$. Note what happens when we write each of the given monomials as a product of the greatest common factor and a second monomial:

$$\begin{aligned} 6x^3y^3 &= 3x^2y^3 \cdot 2x \\ 9x^2y^5 &= 3x^2y^3 \cdot 3y \end{aligned}$$

Observe that the set of second monomial factors ($2x$ and $3y$) contain no additional common factors.

Answer: $4xy^2$

□

You Try It!

EXAMPLE 6. Find the greatest common factor of $12x^4$, $18x^3$, and $30x^2$.

Find the greatest common factor of $6y^3$, $15y^2$, and $9y^5$.

Solution: To find the GCF of $12x^4$, $18x^3$, and $30x^2$, we note that:

1. The greatest common factor (divisor) of 12, 18, and 30 is 6.
2. The monomials $12x^4$, $18x^3$, and $30x^2$ have the variable x in common.
3. The highest power of x in common is x^2.

Thus, the greatest common factor is $\text{GCF}(12x^4, 18x^3, 30x^2) = 6x^2$. Note what happens when we write each of the given monomials as a product of the greatest common factor and a second monomial:

$$\begin{aligned} 12x^4 &= 6x^2 \cdot 2x^2 \\ 18x^3 &= 6x^2 \cdot 3x \\ 30x^2 &= 6x^2 \cdot 5 \end{aligned}$$

Observe that the set of second monomial factors ($2x^2$, $3x$, and 5) contain no additional common factors.

Answer: $3y^2$

Factor Out the GCF

In Chapter 5, we multiplied a monomial and polynomial by distributing the monomial times each term in the polynomial.

$$\begin{aligned} 2x(3x^2 + 4x - 7) &= 2x \cdot 3x^2 + 2x \cdot 4x - 2x \cdot 7 \\ &= 6x^3 + 8x^2 - 14x \end{aligned}$$

In this section we reverse that multiplication process. We present you with the final product and ask you to bring back the original multiplication problem. In the case $6x^3 + 8x^2 - 14x$, the greatest common factor of $6x^3$, $8x^2$, and $14x$ is $2x$. We then use the distributive property to factor out $2x$ from each term of the polynomial.

$$\begin{aligned} 6x^3 + 8x^2 - 14x &= 2x \cdot 3x^2 + 2x \cdot 4x - 2x \cdot 7 \\ &= 2x(3x^2 + 4x - 7) \end{aligned}$$

Factoring. Factoring is “unmultiplying.” You are given the product, then asked to find the original multiplication problem.

First rule of factoring. If the terms of the given polynomial have a greatest common factor (GCF), then factor out the GCF.

Let’s look at a few examples that factor out the GCF.

You Try It!

Factor: $9y^2 - 15y + 12$

EXAMPLE 7. Factor: $6x^2 + 10x + 14$

Solution: The greatest common factor (GCF) of $6x^2$, $10x$ and 14 is 2. Factor out the GCF.

$$\begin{aligned} 6x^2 + 10x + 14 &= 2 \cdot 3x^2 + 2 \cdot 5x + 2 \cdot 7 \\ &= 2(3x^2 + 5x + 7) \end{aligned}$$

Checking your work. Every time you factor a polynomial, remultiply to check your work.

Check: Multiply. Distribute the 2.

$$\begin{aligned} 2(3x^2 + 5x + 7) &= 2 \cdot 3x^2 + 2 \cdot 5x + 2 \cdot 7 \\ &= 6x^2 + 10x + 14 \end{aligned}$$

That's the original polynomial, so we factored correctly.

Answer: $3(3y^2 - 5y + 4)$

You Try It!

EXAMPLE 8. Factor: $12y^5 - 32y^4 + 8y^2$

Factor: $8x^6 + 20x^4 - 24x^3$

Solution: The greatest common factor (GCF) of $12y^5$, $32y^4$ and $8y^2$ is $4y^2$. Factor out the GCF.

$$\begin{aligned} 12y^5 - 32y^4 + 8y^2 &= 4y^2 \cdot 3y^3 - 4y^2 \cdot 8y^2 + 4y^2 \cdot 2 \\ &= 4y^2(3y^3 - 8y^2 + 2) \end{aligned}$$

Check: Multiply. Distribute the monomial $4y^2$.

$$\begin{aligned} 4y^2(3y^3 - 8y^2 + 2) &= 4y^2 \cdot 3y^3 - 4y^2 \cdot 8y^2 + 4y^2 \cdot 2 \\ &= 12y^5 - 32y^4 + 8y^2 \end{aligned}$$

That's the original polynomial. We have factored correctly.

Answer: $4x^3(2x^3 + 5x - 6)$

You Try It!

EXAMPLE 9. Factor: $12a^3b + 24a^2b^2 + 12ab^3$

Factor: $15s^2t^4 + 6s^3t^2 + 9s^2t^2$

Solution: The greatest common factor (GCF) of $12a^3b$, $24a^2b^2$ and $12ab^3$ is $12ab$. Factor out the GCF.

$$\begin{aligned} 12a^3b + 24a^2b^2 + 12ab^3 &= 12ab \cdot a^2 - 12ab \cdot 2ab + 12ab \cdot b^2 \\ &= 12ab(a^2 + 2ab + b^2) \end{aligned}$$

Check: Multiply. Distribute the monomial $12ab$.

$$\begin{aligned} 12ab(a^2 + 2ab + b^2) &= 12ab \cdot a^2 - 12ab \cdot 2ab + 12ab \cdot b^2 \\ &= 12a^3b + 24a^2b^2 + 12ab^3 \end{aligned}$$

Answer: $3s^2t^2(5t^2 + 2s + 3)$

That's the original polynomial. We have factored correctly. □

Speeding Things Up a Bit

Eventually, after showing your work on a number of examples such as those in Examples 7, 8, and 9, you'll need to learn how to perform the process mentally.

You Try It!

Factor: $18p^5q^4 - 30p^4q^5 + 42p^3q^6$

EXAMPLE 10. Factor each of the following polynomials: (a) $24x + 32$, (b) $5x^3 - 10x^2 - 10x$, and (c) $2x^4y + 2x^3y^2 - 6x^2y^3$.

Solution: In each case, factor out the greatest common factor (GCF):

a) The GCF of $24x$ and 32 is 8. Thus, $24x + 32 = 8(3x + 4)$

b) The GCF of $5x^3$, $10x^2$, and $10x$ is $5x$. Thus:

$$5x^3 - 10x^2 - 10x = 5x(x^2 - 2x - 2)$$

c) The GCF of $2x^4y$, $2x^3y^2$, and $6x^2y^3$ is $2x^2y$. Thus:

$$2x^4y + 2x^3y^2 - 6x^2y^3 = 2x^2y(x^2 + xy - 3y^2)$$

As you speed things up by mentally factoring out the GCF, it is even more important that you check your results. The check can also be done mentally. For example, in checking the third result, mentally distribute $2x^2y$ times each term of $x^2 + xy - 3y^2$. Multiplying $2x^2y$ times the first term x^2 produces $2x^4y$, the first term in the original polynomial.

$$2x^2y\,(x^2 + xy - 3y^2) = 2x^4y + 2x^3y^2 - 6x^2y^3$$

Continue in this manner, mentally checking the product of $2x^2y$ with each term of $x^2 + xy - 3y^2$, making sure that each result agrees with the corresponding term of the original polynomial.

Answer: $6p^3q^4(3p^2 - 5pq + 7q^2)$

□

Remember that the distributive property allows us to pull the GCF out in front of the expression or to pull it out in back. In symbols:

$$ab + ac = a(b + c) \qquad \text{or} \qquad ba + ca = (b + c)a$$

You Try It!

Factor: $3x^2(4x-7)+8(4x-7)$

EXAMPLE 11. Factor: $2x(3x+2)+5(3x+2)$

Solution: In this case, the greatest common factor (GCF) is $3x+2$.

$$\begin{aligned}2x(3x+2)+5(3x+2) &= 2x\cdot(3x+2)+5\cdot(3x+2)\\ &= (2x+5)(3x+2)\end{aligned}$$

Because of the commutative property of multiplication, it is equally valid to pull the GCF out in front.

$$\begin{aligned}2x(3x+2)+5(3x+2) &= (3x+2)\cdot 2x+(3x+2)\cdot 5\\ &= (3x+2)(2x+5)\end{aligned}$$

Note that the order of factors differs from the first solution, but because of the commutative property of multiplication, the order does not matter. The answers are the same.

Answer: $(3x^2+8)(4x-7)$

□

You Try It!

Factor: $24m(m-2n)+20(m-2n)$

EXAMPLE 12. Factor: $15a(a+b)-12(a+b)$

Solution: In this case, the greatest common factor (GCF) is $3(a+b)$.

$$\begin{aligned}15a(a+b)-12(a+b) &= 3(a+b)\cdot 5a-3(a+b)\cdot 4\\ &= 3(a+b)(5a-4)\end{aligned}$$

Alternate solution: It is possible that you might fail to notice that 15 and 12 are divisible by 3, factoring out only the common factor $a+b$.

$$\begin{aligned}15a(a+b)-12(a+b) &= 15a\cdot(a+b)-12\cdot(a+b)\\ &= (15a-12)(a+b)\end{aligned}$$

However, you now need to notice that you can continue, factoring out 3 from both $15a$ and 12.

$$= 3(5a-4)(a+b)$$

Note that the order of factors differs from the first solution, but because of the commutative property of multiplication, the order does not matter. The answers are the same.

Answer: $4(6m+5)(m-2n)$

□

Factoring by Grouping

The final factoring skill in this section involves four-term expressions. The technique for factoring a four-term expression is called *factoring by grouping.*

You Try It!

Factor by grouping:
$x^2 - 6x + 2x - 12$

EXAMPLE 13. Factor by grouping: $x^2 + 8x + 3x + 24$

Solution: We "group" the first and second terms, noting that we can factor an x out of both of these terms. Then we "group" the third and fourth terms, noting that we can factor 3 out of both of these terms.

$$x^2 + 8x + 3x + 24 = x\,(x + 8) + 3\,(x + 8)$$

Now we can factor $x + 8$ out of both of these terms.

$$= (x + 3)(x + 8)$$

Answer: $(x + 2)(x - 6)$

□

Let's try a grouping that contains some negative signs.

You Try It!

Factor by grouping:
$x^2 - 5x - 4x + 20$

EXAMPLE 14. Factor by grouping: $x^2 + 4x - 7x - 28$

Solution: We "group" the first and second terms, noting that we can factor x out of both of these terms. Then we "group" the third and fourth terms, then try to factor a 7 out of both these terms.

$$x^2 + 4x - 7x - 28 = x\,(x + 4) + 7\,(-x - 4)$$

This does *not* lead to a common factor. Let's try again, this time factoring a -7 out of the third and fourth terms.

$$x^2 + 4x - 7x - 28 = x\,(x + 4) - 7\,(x + 4)$$

That worked! We now factor out a common factor $x + 4$.

$$= (x - 7)(x + 4)$$

Answer: $(x - 4)(x - 5)$

□

Let's increase the size of the numbers a bit.

You Try It!

EXAMPLE 15. Factor by grouping: $6x^2 - 8x + 9x - 12$

Solution: Note that we can factor $2x$ out of the first two terms and 3 out of the second two terms.

$$6x^2 - 8x + 9x - 12 = 2x\,(3x - 4) + 3\,(3x - 4)$$

Now we have a common factor $3x - 4$ which we can factor out.

$$= (2x + 3)(3x - 4)$$

Factor by grouping: $15x^2 + 9x + 10x + 6$

Answer: $(3x + 2)(5x + 3)$

□

As the numbers get larger and larger, you need to factor out the GCF from each grouping. If not, you won't get a common factor to finish the factoring.

You Try It!

EXAMPLE 16. Factor by grouping: $24x^2 - 32x - 45x + 60$

Solution: Suppose that we factor $8x$ out of the first two terms and -5 out of the second two terms.

$$24x^2 - 32x - 45x + 60 = 8x\,(3x - 4) - 5\,(9x - 12)$$

That did not work, as we don't have a common factor to complete the factoring process. However, note that we can still factor out a 3 from $9x - 12$. As we've already factored out a 5, and now we see can factor out an additional 3, this means that we should have factored out 3 times 5, or 15, to begin with. Let's start again, only this time we'll factor 15 out of the second two terms.

$$24x^2 - 32x - 45x + 60 = 8x\,(3x - 4) - 15\,(3x - 4)$$

Beautiful! We can now factor out $3x - 4$.

$$= (8x - 15)(3x - 4)$$

Factor by grouping: $36x^2 - 84x + 15x - 35$

Answer: $(12x + 5)(3x - 7)$

□

1.1 Exercises

In Exercises 1-6, list all positive divisors of the given number, in order, from smallest to largest.

1. 42

2. 60

3. 44

4. 85

5. 51

6. 63

In Exercises 7-12, list all common positive divisors of the given numbers, in order, from smallest to largest.

7. 36 and 42

8. 54 and 30

9. 78 and 54

10. 96 and 78

11. 8 and 76

12. 99 and 27

In Exercises 13-18, state the greatest common divisor of the given numbers.

13. 76 and 8

14. 84 and 60

15. 32 and 36

16. 64 and 76

17. 24 and 28

18. 63 and 27

In Exercises 19-24, use prime factorization to help calculate the greatest common divisor of the given numbers.

19. 600 and 1080

20. 150 and 120

21. 1800 and 2250

22. 540 and 150

23. 600 and 450

24. 4500 and 1800

In Exercises 25-36, find the greatest common factor of the given expressions.

25. $16b^4$ and $56b^9$

26. $28s^2$ and $36s^4$

27. $35z^2$ and $49z^7$

28. $24w^3$ and $30w^8$

29. $56x^3y^4$ and $16x^2y^5$

30. $35b^5c^3$ and $63b^4c^4$

31. $24s^4t^5$ and $16s^3t^6$

32. $10v^4w^3$ and $8v^3w^4$

33. $18y^7$, $45y^6$, and $27y^5$

34. $8r^7$, $24r^6$, and $12r^5$

35. $9a^6$, $6a^5$, and $15a^4$

36. $15a^5$, $24a^4$, and $24a^3$

In Exercises 37-52, factor out the GCF from each of the given expressions.

37. $25a^2 + 10a + 20$

38. $40c^2 + 15c + 40$

39. $35s^2 + 25s + 45$

40. $45b^2 + 20b + 35$

41. $16c^3 + 32c^2 + 36c$

42. $12b^3 + 12b^2 + 18b$

43. $42s^3 + 24s^2 + 18s$

44. $36y^3 + 81y^2 + 36y$

45. $35s^7 + 49s^6 + 63s^5$

46. $35s^7 + 56s^6 + 56s^5$

47. $14b^7 + 35b^6 + 56b^5$

48. $45x^5 + 81x^4 + 45x^3$

49. $54y^5z^3 + 30y^4z^4 + 36y^3z^5$

50. $42x^4y^2 + 42x^3y^3 + 54x^2y^4$

51. $45s^4t^3 + 40s^3t^4 + 15s^2t^5$

52. $20v^6w^3 + 36v^5w^4 + 28v^4w^5$

In Exercises 53-60, factor out the GCF from each of the given expressions.

53. $7w(2w - 3) - 8(2w - 3)$

54. $5s(8s - 1) + 4(8s - 1)$

55. $9r(5r - 1) + 8(5r - 1)$

56. $5c(4c - 7) + 2(4c - 7)$

57. $48a(2a + 5) - 42(2a + 5)$

58. $40v(7v - 4) + 72(7v - 4)$

59. $56a(2a - 1) - 21(2a - 1)$

60. $48r(5r + 3) - 40(5r + 3)$

In Exercises 61-68, factor by grouping. Do **not** simplify the expression before factoring.

61. $x^2 + 2x - 9x - 18$

62. $x^2 + 6x - 9x - 54$

63. $x^2 + 3x + 6x + 18$

64. $x^2 + 8x + 7x + 56$

65. $x^2 - 6x - 3x + 18$

66. $x^2 - 3x - 9x + 27$

67. $x^2 - 9x + 3x - 27$

68. $x^2 - 2x + 7x - 14$

In Exercises 69-76, factor by grouping. Do **not** simplify the expression before factoring.

69. $8x^2 + 3x - 56x - 21$

70. $4x^2 + 9x - 32x - 72$

71. $9x^2 + 36x - 5x - 20$

72. $7x^2 + 14x - 8x - 16$

73. $6x^2 - 7x - 48x + 56$

74. $8x^2 - 7x - 72x + 63$

75. $2x^2 + 12x + 7x + 42$

76. $7x^2 + 28x + 9x + 36$

1.1 Answers

1. $\{1, 2, 3, 6, 7, 14, 21, 42\}$

3. $\{1, 2, 4, 11, 22, 44\}$

5. $\{1, 3, 17, 51\}$

7. $\{1, 2, 3, 6\}$

9. $\{1, 2, 3, 6\}$

11. $\{1, 2, 4\}$

13. 4

15. 4

17. 4

19. 120

21. 450

23. 150

25. $8b^4$

27. $7z^2$

29. $8x^2y^4$

31. $8s^3t^5$

33. $9y^5$

35. $3a^4$

37. $5(5a^2 + 2a + 4)$

39. $5(7s^2 + 5s + 9)$

41. $4c(4c^2 + 8c + 9)$

43. $6s(7s^2 + 4s + 3)$

45. $7s^5(5s^2 + 7s + 9)$

47. $7b^5(2b^2 + 5b + 8)$

49. $6y^3z^3(9y^2 + 5yz + 6z^2)$

51. $5s^2t^3(9s^2 + 8st + 3t^2)$

53. $(7w - 8)(2w - 3)$

55. $(9r + 8)(5r - 1)$

57. $6(2a + 5)(8a - 7)$

59. $7(2a - 1)(8a - 3)$

61. $(x - 9)(x + 2)$

63. $(x + 6)(x + 3)$

65. $(x - 3)(x - 6)$

67. $(x + 3)(x - 9)$

69. $(x - 7)(8x + 3)$

71. $(9x - 5)(x + 4)$

73. $(x - 8)(6x - 7)$

75. $(2x + 7)(x + 6)$

1.2 Solving Nonlinear Equations

We begin by introducing a property that will be used extensively in this and future sections.

The zero product property. If the product of two or more numbers equals zero, then at least one of the numbers must equal zero. That is, if

$$ab = 0,$$

then

$$a = 0 \quad \text{or} \quad b = 0.$$

Let's use the *zero product property* to solve a few equations.

You Try It!

Solve for x:
$(x-7)(x-2)=0$

Answer: 7, 2

EXAMPLE 1. Solve for x: $(x+3)(x-5)=0$

Solution: The product of two factors equals zero.

$$(x+3)(x-5)=0$$

Hence, at least one of the factors must equal zero. Using the zero product property, set each factor equal to zero, then solve the resulting equations for x.

$$\begin{aligned} x+3&=0 &\quad \text{or} \quad x-5&=0 \\ x&=-3 & x&=5 \end{aligned}$$

Hence, the solutions are $x=-3$ and $x=5$

Check: Check that each solution satisfies the original equation.

Substitute -3 for x:

$$\begin{aligned} (x+3)(x-5)&=0 \\ (-3+3)(-3-5)&=0 \\ (0)(-8)&=0 \\ 0&=0 \end{aligned}$$

Substitute 5 for x:

$$\begin{aligned} (x+3)(x-5)&=0 \\ (5+3)(5-5)&=0 \\ (8)(0)&=0 \\ 0&=0 \end{aligned}$$

Because each check produces a true statement, both $x=-3$ and $x=5$ are solutions of $(x+3)(x-5)=0$.

□

The zero product property also works equally well if more than two factors are present. For example, if $abc=0$, then either $a=0$ or $b=0$ or $c=0$. Let's use this idea in the next example.

You Try It!

Solve for x:
$6x(x+4)(5x+1)=0$

EXAMPLE 2. Solve for x: $x(2x+9)(3x-5)=0$

Solution: The product of three factors equals zero.

$$x(2x+9)(3x-5)=0$$

Using the zero product property, set each factor equal to zero, then solve the resulting equations for x.

$$\begin{aligned} x &= 0 \quad \text{or} & 2x+9 &= 0 \quad \text{or} & 3x-5 &= 0 \\ & & 2x &= -9 & 3x &= 5 \\ & & x &= -\frac{9}{2} & x &= \frac{5}{3} \end{aligned}$$

Hence, the solutions are $x=0$, $x=-9/2$, and $x=5/3$. We encourage the reader to check the solution.

Answer: $0, -4, -1/5$

□

Linear versus Nonlinear

All of the equations solved in previous chapters were examples of what are called *linear equations*. If the highest power of the variable we are solving for is one, then the graphs involved are lines. Hence the term, *linear equation*. However, if the power on the variable we are solving for exceeds one, then the graphs involved are curves. Hence the term, *nonlinear equation*. In this chapter we will learn how to solve *nonlinear equations* involving polynomials. However, let's first make sure we can recognize the difference between a linear and a nonlinear equation.

Linear versus Nonlinear. Use the following conditions to determine if an equation is linear or nonlinear.

1. If the highest power of the variable we are solving for is one, then the equation is *linear*.
2. If the highest power of the variable we are solving for is larger than one, then the equation is *nonlinear*.

You Try It!

Classify the following equation as linear or nonlinear: $2x=x^3-4$

EXAMPLE 3. If the instruction is "solve for x," classify each of the following equations as linear or nonlinear.

a) $3x - 5 = 4 - 7x$ b) $x^2 = 8x$

Solution: Because the instruction is "solve for x," to determine whether the equation is linear or nonlinear, we identify the largest power of x present in the equation.

a) The highest power of x present in the equation $3x - 5 = 4 - 7x$ is one. Hence, this equation is *linear*.

b) The equation $x^2 = 8x$ contains a power of x higher than one (it contains an x^2). Hence, this equation is *nonlinear*.

Answer: nonlinear

Now that we can classify equations as either *linear* or *nonlinear*, let's introduce strategies for solving each type, the first of which should already be familiar.

Strategy for solving a linear equation. If an equation is *linear*, start the solution process by moving all terms containing the variable you are solving for to one side of the equation, then move all terms that do not contain the variable you are solving for to the other side of the equation.

You Try It!

Solve for x:
$2x + 4 = 6x + 3$

EXAMPLE 4. Solve for x: $3x - 5 = 4 - 7x$

Solution: Because the instruction is "solve for x" and we note that the largest power of x present is one, the equation $3x - 5 = 4 - 7x$ is linear. Hence, the strategy is to move all terms containing x to one side of the equation, then move all the remaining terms to the other side of the equation.

$$3x - 5 = 4 - 7x \quad \text{Original equation.}$$
$$3x - 5 + 7x = 4 \quad \text{Add } 7x \text{ to both sides.}$$
$$3x + 7x = 4 + 5 \quad \text{Add 5 to both sides.}$$

Note how we have succeeded in moving all terms containing x to one side of the equation and all terms that do not contain x to the other side of the equation.

$$10x = 9 \quad \text{Simplify both sides.}$$
$$x = \frac{9}{10} \quad \text{Divide both sides by 10.}$$

Hence, the solution of $3x - 5 = 4 - 7x$ is $x = 9/10$. Readers are encouraged to check this solution.

Answer: 1/4

The situation is much different when the equation is nonlinear.

Strategy for solving a nonlinear equation. If an equation is *nonlinear*, first move everything to one side of the equation, making one side of the equation equal to zero. Continue the solution process by factoring and applying the *zero product property.*

You Try It!

Solve for x: $x^2 = -5x$

EXAMPLE 5. Solve for x: $x^2 = 8x$

Solution: Because the instruction is "solve for x," and the highest power of x is larger than one, the equation $x^2 = 8x$ is *nonlinear*. Hence, the strategy requires that we move all terms to one side of the equation, making one side zero.

$$x^2 = 8x \qquad \text{Original equation.}$$

$$x^2 - 8x = 0 \qquad \text{Subtract } 8x \text{ from both sides.}$$

Note how we have succeeded in moving all terms to one side of the equation, making one side equal to zero. To finish the solution, we factor out the GCF on the left-hand side.

$$x(x-8) = 0 \qquad \text{Factor out the GCF.}$$

Note that we now have a product of two factors that equals zero. By the zero product property, either the first factor is zero or the second factor is zero.

$$x = 0 \quad \text{or} \quad x - 8 = 0$$

$$x = 8$$

Hence, the solutions are $x = 0$ and $x = 8$.

Check: Check that each solution satisfies the original equation.

Substitute 0 for x:

$$x^2 = 8x$$
$$(0)^2 = 8(0)$$
$$0 = 0$$

Subtitute 8 for x:

$$x^2 = 8x$$
$$(8)^2 = 8(8)$$
$$64 = 64$$

Note that both results are true statements, guaranteeing that both $x = 0$ and $x = 8$ are solutions of $x^2 = 8x$.

Answer: 0, -5

□

Warning! The following is incorrect! Consider what would happen if we divided both sides of the equation $x^2 = 8x$ in Example 5 by x:

$$\begin{aligned} x^2 &= 8x \\ \frac{x^2}{x} &= \frac{8x}{x} \\ x &= 8 \end{aligned}$$

Note that we have lost the second answer found in Example 5, $x = 0$. This example demonstrates that you should **never divide by the variable you are solving for!** If you do, and cancellation occurs, you will lose answers.

Let's try solving a nonlinear equation that requires factoring by grouping.

You Try It!

Solve for x:
$5x^2 - 20x - 4x + 16 = 0$

EXAMPLE 6. Solve for x: $6x^2 + 9x - 8x - 12 = 0$

Solution: Because we are solving for x and there is a power of x larger than one, this equation is nonlinear. Hence, the first step is to move everything to one side of the equation, making one side equal to zero. Well that's already done, so let's factor the left-hand side by grouping. Note that we can factor $3x$ out of the first two terms and -4 out of the second two terms.

$$\begin{aligned} 6x^2 + 9x - 8x - 12 &= 0 \\ 3x(2x+3) - 4(2x+3) &= 0 \end{aligned}$$

Factor out the common factor $2x + 3$.

$$(3x - 4)(2x + 3) = 0$$

We now have a product of two factors that equals zero. Use the zero product property to write:

$$\begin{aligned} 3x - 4 &= 0 & \quad\text{or}\quad && 2x + 3 &= 0 \\ 3x &= 4 & && 2x &= -3 \\ x &= \frac{4}{3} & && x &= -\frac{3}{2} \end{aligned}$$

Hence, the solutions are $x = 4/3$ and $x = -3/2$.

Check. Let's use the graphing calculator to check the solution $x = 4/3$. First, store the solution $4/3$ in the variable **X** using the following keystrokes (see the first image in Figure 1.1.

4 ÷ 3 STO▷ X,T,θ,n ENTER

Next, enter the left-hand side of the equation $6x^2 + 9x - 8x - 12 = 0$ using the following keystrokes. Note that the result in the second image in Figure 1.1 indicates that the expression $6x^2 + 9x - 8x - 12$ equals zero when $x = 4/3$.

6 × X,T,θ,n ∧ 2 + 9 × X,T,θ,n −

8 × X,T,θ,n − 1 2 ENTER

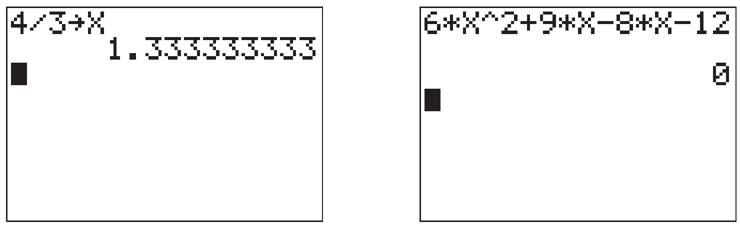


Figure 1.1: Checking the solution $x = 4/3$.

Therefore, the solution $x = 4/3$ checks. Readers are encouraged to use their graphing calculators to check the second solution, $x = -3/2$.

Answer: 4/5, 4

□

Using the Graphing Calculator

In this section we will employ two different calculator routines to find the solution of a nonlinear equation. Before picking up the calculator, let's first use an algebraic method to solve the equation $x^2 = -5x$. The equation is nonlinear, so the first step is to move everything to one side of the equation, making one side equal to zero.

$x^2 = -5x$	Nonlinear. Make one side zero.
$x^2 + 5x = 0$	Add $5x$ to both sides.
$x(x + 5) = 0$	Factor out the GCF.

Use the zero product property, setting each factor equal to zero, then solving the resulting equations for x.

$$x = 0 \quad \text{or} \quad x + 5 = 0$$
$$x = -5$$

Hence, the solutions are $x = 0$ and $x = -5$.

We'll now use the calculator to find the solutions of $x^2 = -5x$. The first technique employs the **5:intersect** routine on the calculator's CALC menu.

You Try It!

EXAMPLE 7. Use the **5:intersect** utility on the graphing calculator to solve the equation $x^2 = -5x$ for x.

Use the **5:intersect** utility on the graphing calculator to solve the equation $x^2 = 4x$ for x.

Solution: Load the left-hand side of $x^2 = -5x$ in **Y1** and the right-hand side in **Y2** (see Figure 1.2). Selecting **6:ZStandard** from the ZOOM menu produces the graphs shown in the image on the right in Figure 1.2.

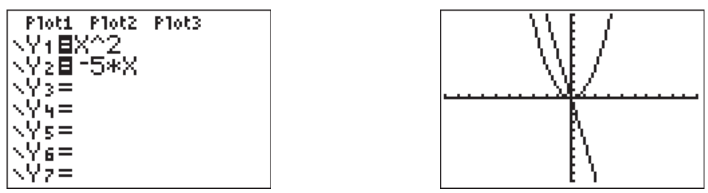


Figure 1.2: Sketch the graphs of each side of the equation $x^2 = -5x$.

Note that the graph of $y = x^2$ is a parabola that opens upward, with vertex (turning point) at the origin. This graph reveals why the equation $x^2 = -5x$ is called a *nonlinear* equation (not all the graphs involved are lines). Next, the graph of $y = -5x$ is a line with slope -5 and y-intercept at the origin.

The two graphs obviously intersect at the origin, but it also appears that there may be another point of intersection that is off the screen. Let's increase **Ymax** in an attempt to reveal the second point of intersection. After some experimentation, the settings shown in the first image in Figure 1.3 reveal both points of intersection. Pushing the GRAPH button produces the image on the right in Figure 1.3.

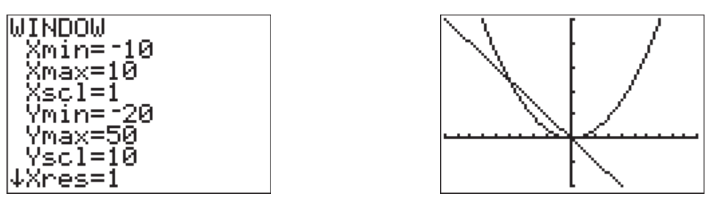


Figure 1.3: Adjust the WINDOW parameters to reveal both points of inter-section.

To find the solutions of the equation $x^2 = -5x$, we must find the coordinates of the points where the graphs of $y = x^2$ and $y = -5x$ intersect. The x-coordinate of each point of intersection will be a solution of the equation $x^2 = -5x$.

- Start by selecting **5:intersect** from the CALC menu. When prompted for the "First curve?", press ENTER. When prompted for the "Second curve?", press ENTER. When prompted for a "Guess," press ENTER. The result is the point $(0, 0)$ shown in the image on the left in Figure 1.4.
- Repeat the process a second time. Select **5:intersect** from the CALC menu. When prompted for the "First curve?", press ENTER. When prompted for the "Second curve?", press ENTER. When prompted for a "Guess," use the left-arrow key to move the cursor closer to the leftmost point of intersection, then press ENTER. The result is the point $(-5, 25)$ shown in the image on the right in Figure 1.4.

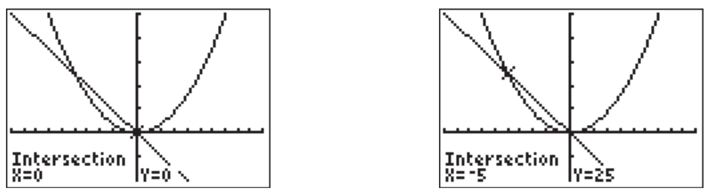
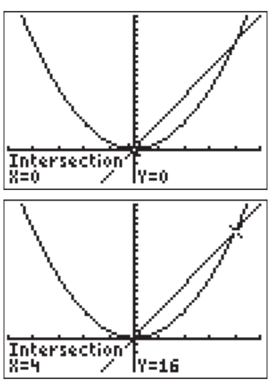


Figure 1.4: Use the **5:intersect** utility to find the points of intersection.

Answer:



Reporting the solution on your homework: Duplicate the image in your calculator's viewing window on your homework page. Use a ruler to draw all lines, but freehand any curves.

- Label the horizontal and vertical axes with x and y, respectively (see Figure 1.5).
- Place your WINDOW parameters at the end of each axis (see Figure 1.5).
- Label each graph with its equation (see Figure 1.5).
- Drop dashed vertical lines through each point of intersection. Shade and label the x-values of the points where the dashed vertical line crosses the x-axis. These are the solutions of the equation $x^2 = -5x$ (see Figure 1.5).

Hence, the solutions of $x^2 = -5x$ are $x = -5$ and $x = 0$. Note now these match the solutions found using the algebraic technique.

□

Before demonstrating a second graphing calculator technique for solving nonlinear equations, let's take a moment to recall the definition of a *zero* of a function, which was first presented in Chapter 5, Section 3.

Zeros and x-intercepts. The points where the graph of f crosses the x-axis are called the *x-intercepts* of the graph of f. The x-value of each x-intercept is called a *zero* of the function f.

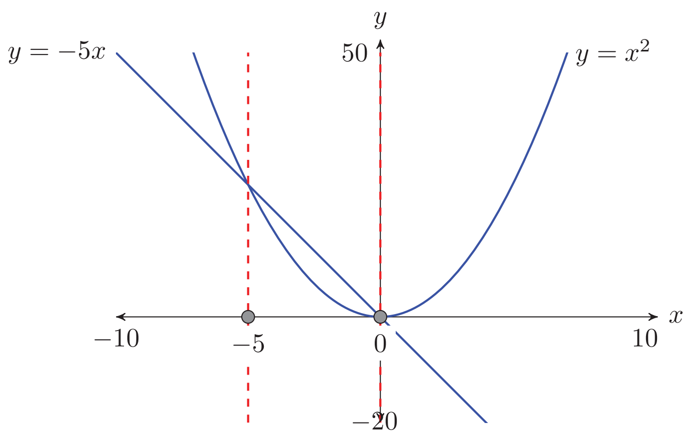


Figure 1.5: Reporting your graphical solution on your homework.

We'll now employ the **2:zero** utility from the CALC menu to find the solutions of the equation $x^2 = -5x$.

You Try It!

Use the **2:zero** utility on the graphing calculator to solve the equation $x^2 = 4x$ for x.

EXAMPLE 8. Use the **2:zero** utility on the graphing calculator to solve the equation $x^2 = -5x$ for x.

Solution: First, make one side of the equation equal to zero.

$$\begin{aligned} x^2 &= -5x && \text{Make one side zero.} \\ x^2 + 5x &= 0 && \text{Add } 5x \text{ to both sides.} \end{aligned}$$

To determine the values of x that make $x^2 + 5x = 0$, we must locate the points where the graph of $f(x) = x^2 + 5x$ crosses the x-axis. These points are the x-intercepts of the graph of f and the x-values of these points are the zeros of the function f.

Load the function $f(x) = x^2 + 5x$ in **Y1**, then select **6:ZStandard** to produce the image in Figure 1.6. Note that the graph of f has two x-intercepts, and the x-values of each of these points are the zeros of the function f.

It's often easier to find the solutions of a nonlinear equation by making one side zero and identifying where the graph of the resulting function crosses the x-axis.

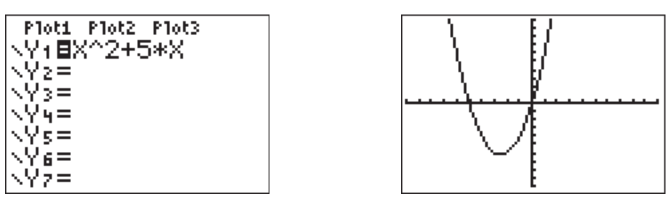


Figure 1.6: Sketch the graph of $p(x) = x^2 + 5\,x$.

Select **2:zero** from the CALC menu (see Figure 1.7).

- The calculator responds by asking for a "Left Bound?" Use the left-arrow key to move the cursor so that it lies to the left of the x-intercept near $(-5, 0)$ (see the second image in Figure 1.7), then press the ENTER key.
- The calculator responds by asking for a "Right Bound?" Move the cursor so that is slightly to the right of the x-intercept near $(-5, 0)$ (see the third image Figure 1.7), then press the ENTER key.

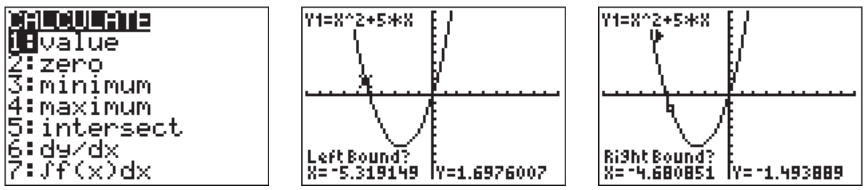


Figure 1.7: Setting the left and right bounds when using the **2:zero** utility to find the x-intercepts of the graph of $f(x) = x^2 + 5x$.

- The calculator responds by asking for a "Guess?" Note the two triangular marks near the top of the viewing window in the first image in Figure 1.8 that mark the left- and right-bounds. As long as you place the cursor so that the x-value of the cursor location lies between these two marks, you've made a valid guess. Because the cursor already lies between these two marks, we usually leave it where it is and press the ENTER key.

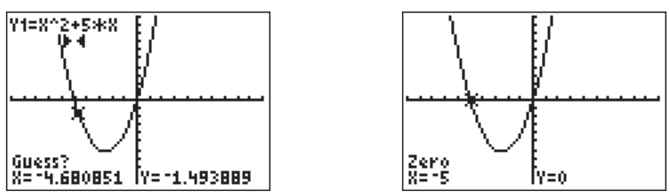


Figure 1.8: Setting the right bound and making a guess.

After making your guess and pressing the ENTER key, the calculator proceeds to find an approximation of the x-intercept that lies between the left- and right-bounds previously marked (see the second image in Figure 1.8). Hence, this x-intercept is $(-5, 0)$, making -5 a zero of $f(x) = x^2 + 5x$ and a solution of the equation $x^2 + 5x = 0$.

We'll leave it to our readers to repeat the **2:zero** process to find the second zero at the origin.

Reporting the solution on your homework: Duplicate the image in your calculator's viewing window on your homework page. Use a ruler to draw all lines, but freehand any curves.

- Label the horizontal and vertical axes with x and y, respectively (see Figure 1.9).
- Place your WINDOW parameters at the end of each axis (see Figure 1.9).
- Label each graph with its equation (see Figure 1.9).
- Drop dashed vertical lines through each x-intercept. Shade and label the x-values of each x-intercept. These are the solutions of the equation $x^2 = -5x$ (see Figure 1.9).

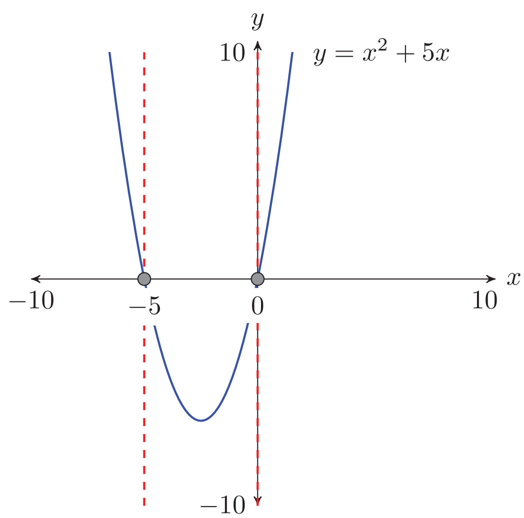
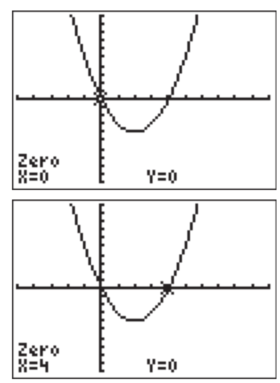


Figure 1.9: Reporting your graphical solution on your homework.

Answer:



Hence, the solutions of $x^2 = -5x$ are $x = -5$ and $x = 0$. Note how nicely this agrees with the solutions found using the algebraic technique and the solutions found using the **5:intersect** utility in Example 7.

1.2 Exercises

In Exercises 1-8, solve the given equation for x.

1. $(9x+2)(8x+3)=0$

2. $(2x-5)(7x-4)=0$

3. $x(4x+7)(9x-8)=0$

4. $x(9x-8)(3x+1)=0$

5. $-9x(9x+4)=0$

6. $4x(3x-6)=0$

7. $(x+1)(x+6)=0$

8. $(x-4)(x-1)=0$

In Exercises 9-18, given that you are solving for x, state whether the given equation is linear or nonlinear. **Do not solve the equation.**

9. $x^2+7x=9x+63$

10. $x^2+9x=4x+36$

11. $6x-2=5x-8$

12. $-5x+5=-6x-7$

13. $7x^2=-2x$

14. $4x^2=-7x$

15. $3x^2+8x=-9$

16. $5x^2-2x=-9$

17. $-3x+6=-9$

18. $8x-5=3$

In Exercises 19-34, solve each of the given equations for x.

19. $3x+8=9$

20. $3x+4=2$

21. $9x^2=-x$

22. $6x^2=7x$

23. $3x+9=8x+7$

24. $8x+5=6x+4$

25. $8x^2=-2x$

26. $8x^2=18x$

27. $9x+2=7$

28. $3x+2=6$

29. $9x^2=6x$

30. $6x^2=-14x$

31. $7x^2=-4x$

32. $7x^2=-9x$

33. $7x+2=4x+7$

34. $4x+3=2x+8$

In Exercises 35-50, factor by grouping to solve each of the given equations for x.

35. $63x^2 + 56x + 54x + 48 = 0$

36. $27x^2 + 36x + 6x + 8 = 0$

37. $16x^2 - 18x + 40x - 45 = 0$

38. $42x^2 - 35x + 54x - 45 = 0$

39. $45x^2 + 18x + 20x + 8 = 0$

40. $18x^2 + 21x + 30x + 35 = 0$

41. $x^2 + 10x + 4x + 40 = 0$

42. $x^2 + 11x + 10x + 110 = 0$

43. $x^2 + 6x - 11x - 66 = 0$

44. $x^2 + 6x - 2x - 12 = 0$

45. $15x^2 - 24x + 35x - 56 = 0$

46. $12x^2 - 10x + 54x - 45 = 0$

47. $x^2 + 2x + 9x + 18 = 0$

48. $x^2 + 8x + 4x + 32 = 0$

49. $x^2 + 4x - 8x - 32 = 0$

50. $x^2 + 8x - 5x - 40 = 0$

In Exercises 51-54, perform each of the following tasks:

i) Use a strictly algebraic technique to solve the given equation.

ii) Use the **5:intersect** utility on your graphing calculator to solve the given equation. Report the results found using graphing calculator as shown in Example 7.

51. $x^2 = -4x$

52. $x^2 = 6x$

53. $x^2 = 5x$

54. $x^2 = -6x$

In Exercises 55-58, perform each of the following tasks:

i) Use a strictly algebraic technique to solve the given equation.

ii) Use the **2:zero** utility on your graphing calculator to solve the given equation. Report the results found using graphing calculator as shown in Example 8.

55. $x^2 + 7x = 0$

56. $x^2 - 8x = 0$

57. $x^2 - 3x = 0$

58. $x^2 + 2x = 0$

1.2 Answers

1. $x = -\frac{2}{9}, -\frac{3}{8}$

3. $x = 0, -\frac{7}{4}, \frac{8}{9}$

11. Linear

13. Nonlinear

15. Nonlinear

17. Linear

19. $\frac{1}{3}$

21. $x = 0, -\frac{1}{9}$

23. $\frac{2}{5}$

25. $x = 0, -\frac{1}{4}$

27. $\frac{5}{9}$

29. $x = 0, \frac{2}{3}$

31. $x = 0, -\frac{4}{7}$

33. $\frac{5}{3}$

5. $x = 0, -\frac{4}{9}$

7. $x = -1, -6$

9. Nonlinear

35. $x = -\frac{6}{7}, -\frac{8}{9}$

37. $x = -\frac{5}{2}, \frac{9}{8}$

39. $x = -\frac{4}{9}, -\frac{2}{5}$

41. $x = -4, -10$

43. $x = 11, -6$

45. $x = -\frac{7}{3}, \frac{8}{5}$

47. $x = -9, -2$

49. $x = 8, -4$

51. $x = -4, 0$

53. $x = 0, 5$

55. $x = -7, 0$

57. $x = 0, 3$

1.3 Factoring $ax^2 + bx + c$ when $a = 1$

In this section we concentrate on learning how to factor trinomials having the form $ax^2 + bx + c$ when $a = 1$. The first task is to make sure that everyone can properly identify the coefficients a, b, and c.

You Try It!

EXAMPLE 1. Compare $x^2 - 8x - 9$ with the form $ax^2 + bx + c$ and identify the coefficients a, b, and c.

Solution: Align the trinomial $x^2 - 8x - 9$ with the standard form $ax^2 + bx + c$, then compare coefficients. Note that the understood coefficient of x^2 is 1.

$$\begin{aligned} &ax^2 + bx + c \\ &1x^2 - 8x - 9 \end{aligned}$$

We see that $a = 1$, $b = -8$, and $c = -9$. Because the leading coefficient is 1, this is the type of trinomial that we will learn how to factor in this section. □

Compare $2x^2 + 5x - 3$ with the form $ax^2 + bx + c$ and identify the coefficients a, b, and c.

Answer: $a = 2$, $b = 5$, $c = -3$

You Try It!

EXAMPLE 2. Compare $-40 + 6x^2 - x$ with the form $ax^2 + bx + c$ and identify the coefficients a, b, and c.

Solution: First, arrange $-40 + 6x^2 - x$ in descending powers of x, then align it with the standard form $ax^2 + bx + c$ and compare coefficients. Note that the understood coefficient of x is -1.

$$\begin{aligned} &ax^2 + bx + c \\ &6x^2 - 1x - 40 \end{aligned}$$

We see that $a = 6$, $b = -1$, and $c = -40$. Because the leading coefficient is 6, we will have to wait until Factoring $ax^2 + bx + c$ when $a \neq 1$ on page 427 before learning how to factor this trinomial. □

Compare $3x + 9 - 7x^2$ with the form $ax^2 + bx + c$ and identify the coefficients a, b, and c.

Answer: $a = -7$, $b = 3$, $c = 9$

In this section, the leading coefficient must equal 1. Our work in this section will focus only on trinomials of the form $x^2 + bx + c$, that is, the form $ax^2 + bx + c$ where $a = 1$.

The *ac*-Method

We are now going to introduce a technique called the *ac-method* (or *ac-test*) for factoring trinomials of the form $ax^2 + bx + c$ when $a = 1$. In the upcoming section Factoring $ax^2 + bx + c$ when $a \neq 1$ on page 427, we will see that this method can also be employed when $a \neq 1$, with one minor exception. But for the remainder of this section, we focus strictly on trinomials whose leading coefficient is 1.

Let's begin by finding the following product:

$$\begin{aligned}
(x+12)(x-4) &= x(x-4) + 12(x-4) && \text{Apply the distributive property.}\\
&= x^2 - 4x + 12x - 48 && \text{Distribute again.}\\
&= x^2 + 8x - 48 && \text{Simplify.}
\end{aligned}$$

Now, can we reverse the process? That is, can we start with $x^2 + 8x - 48$ and place it in its original factored form $(x + 12)(x - 4)$? The answer is yes, if we apply the following procedure.

The *ac*-method. Compare the given polynomial with the standard form $ax^2 + bx + c$, determine the coefficients a, b, and c, then proceed as follows:

1. Multiply the coefficients a and c and determine their product ac. List all the integer pairs whose product equals ac.
2. Circle the pair in the list produced in step 1 whose sum equals b, the coefficient of the middle term of $ax^2 + bx + c$.
3. Replace the middle term bx with a sum of like terms using the circled pair from step 2.
4. Factor by grouping.
5. Check the result using the FOIL shortcut.

Let's follow the steps of the ac-method to factor $x^2 + 8x - 48$.

You Try It!

Factor: $x^2 + 11x + 28$

EXAMPLE 3. Factor: $x^2 + 8x - 48$.

Solution: Compare $x^2+8x-48$ with ax^2+bx+c and identify $a = 1$, $b = 8$, and $c = -48$. Note that the leading coefficient is $a = 1$. Calculate ac. Note that $ac = (1)(-48)$, so $ac = -48$. List all integer pairs whose product is $ac = -48$.

$1, -48$	$-1, 48$
$2, -24$	$-2, 24$
$3, -16$	$-3, 16$
$4, -12$	$-4, 12$
$6, -8$	$-6, 8$

Circle the ordered pair whose sum is $b = 8$.

$1, -48$	$-1, 48$
$2, -24$	$-2, 24$
$3, -16$	$-3, 16$
$4, -12$	$\boxed{-4, 12}$
$6, -8$	$-6, 8$

Replace the middle term $8x$ with a sum of like terms using the circled pair whose sum is 8.

$$x^2 + 8x - 48 = x^2 - 4x + 12x - 48$$

Factor by grouping.

$$\begin{aligned} x^2 + 8x - 48 &= x(x-4) + 12(x-4) \\ &= (x+12)(x-4) \end{aligned}$$

Use the FOIL shortcut to mentally check your answer. To determine the product $(x + 12)(x - 4)$, use these steps:

- Multiply the terms in the "First" positions: x^2.
- Multiply the terms in the "Outer" and "Inner" positions and combine the results mentally: $-4x + 12x = 8x$.
- Multiply the terms in the "Last" positions: -48.

That is:

$$(x+12)(x-4) = \overset{F}{x^2} - \overset{O}{4x} + \overset{I}{12x} - \overset{L}{48}$$

Combining like terms, $(x + 12)(x - 4) = x^2 + 8x - 48$, which is the original trinomial, so our solution checks. Note that if you combine the "Outer" and "Inner" products mentally, the check goes even faster.

Answer: $(x + 4)(x + 7)$

□

Some readers might ask "Is it a coincidence that the circled pair $\boxed{-4, 12}$ seemed to 'drop in place' in the resulting factoriztion $(x + 12)(x - 4)$?" Before we answer that question, let's try another example.

You Try It!

Factor: $x^2 + 10x - 24$

EXAMPLE 4. Factor: $x^2 - 9x - 36$.

Solution: Compare $x^2 - 9x - 36$ with $ax^2 + bx + c$ and note that $a = 1$, $b = -9$, and $c = -36$. Calculate $ac = (1)(-36)$, so $ac = -36$.

At this point, some readers might ask "What if I start listing the ordered pairs and I see the pair I need? Do I need to continue listing the remaining pairs?

The answer is "No." In this case, we start listing the integer pairs whose product is $ac = -36$, but are mindful that we need an integer pair whose sum is $b = -9$. The integer pair 3 and -12 has a product equaling $ac = -36$ and a sum equaling $b = -9$.

$$1, -36$$
$$2, -18$$
$$\boxed{3, -12}$$

Note how we ceased listing ordered pairs the moment we found the pair we needed. Next, replace the middle term $-9x$ with a sum of like terms using the circled pair.

$$x^2 - 9x - 36 = x^2 + 3x - 12x - 36$$

Factor by grouping.

$$\begin{aligned} x^2 - 9x - 36 &= x(x+3) - 12(x+3) \\ &= (x-12)(x+3) \end{aligned}$$

Use the FOIL shortcut to check your answer.

$$\begin{array}{ccccccccc} & & F & & O & & I & & L \\ (x+3)(x-12) & = & x^2 & - & 12x & + & 3x & - & 36 \end{array}$$

Combining like terms, $(x+3)(x-12) = x^2 - 9x - 36$, the original trinomial. Our solution checks.

Answer: $(x+12)(x-2)$

□

Speeding Things Up a Bit

Readers might again ask "Is it a coincidence that the circled pair $\boxed{3, -12}$ seemed to 'drop in place' in the resulting factoriztion $(x-12)(x+3)$?" The answer is "No," it is not a coincidence. Provided the leading coefficient of the trinomial $ax^2 + bx + c$ is $a = 1$, you can always "drop in place" the circled pair in order to arrive at the final factorization, skipping the factoring by grouping.

Some readers might also be asking "Do I really have to list **any** of those ordered pairs if I already recognize the pair I need?" The answer is "No!" If you see the pair you need, drop it in place.

You Try It!

Factor: $x^2 - 12x + 35$

EXAMPLE 5. Factor: $x^2 - 5x - 24$.

Solution: Compare $x^2 - 5x - 24$ with $ax^2 + bx + c$ and note that $a = 1$, $b = -5$, and $c = -24$. Calculate $ac = (1)(-24)$, so $ac = -24$. Now can you think of an integer pair whose product is $ac = -24$ and whose sum is $b = -5$? For some, the required pair just pops into their head: -8 and 3. The product of these two integers is -24 and their sum is -5. "Drop" this pair in place and you are done.

$$x^2 - 5x - 24 = (x-8)(x+3)$$

Use the FOIL shortcut to check your answer.

$$\begin{array}{ccccccccc} & & F & & O & & I & & L \\ (x-8)(x+3) & = & x^2 & + & 3x & - & 8x & - & 24 \end{array}$$

Combining like terms, $(x-8)(x+3) = x^2 - 5x - 24$, the original trinomial. Our solution checks.

Answer: $(x-7)(x-5)$

The "Drop in Place" technique of Example 5 allows us to revise the *ac*-method a bit.

Revised *ac*-method. Compare the given polynomial with the standard form $ax^2 + bx + c$, determine the coefficients a, b, and c, then determine a pair of integers whose product equals ac and whose sum equals b. You then have two options:

1. Write the middle term as a product of like terms using the ordered pair whose product is ac and whose sum is b. Complete the factorization process by factoring by grouping.
2. (**Only works if** $a = 1$.) Simply "drop in place" the ordered pair whose product is ac and whose sum is b to complete the factorization process. *Note: We'll see in Factoring $ax^2 + bx + c$ when $a \neq 1$ on page 427 why this "drop in place" choice does not work when $a \neq 1$.*

Readers are strongly encouraged to check their factorization by determining the product using the FOIL method. If this produces the original trinomial, the factorization is correct.

Nonlinear Equations Revisited

The ability to factor trinomials of the form $ax^2 + bx + c$, where $a = 1$, increases the number of nonlinear equations we are now able to solve.

You Try It!

EXAMPLE 6. Solve the equation $x^2 = 2x + 3$ both algebraically and graphically, then compare your answers.

Solve the equation $x^2 = -3x + 4$ both algebraically and graphically, then compare your answers.

Solution: Because there is a power of x larger than one, the equation is nonlinear. Make one side zero.

$$\begin{aligned} x^2 &= 2x+3 && \text{Original equation.} \\ x^2-2x &= 3 && \text{Subtract } 2x \text{ from both sides.} \\ x^2-2x-3 &= 0 && \text{Subtract 3 from both sides.} \end{aligned}$$

Compare x^2-2x-3 with ax^2+bx+c and note that $a=1$, $b=-2$ and $c=-3$. We need an integer pair whose product is $ac=-3$ and whose sum is $b=-2$. The integer pair 1 and -3 comes to mind. "Drop" these in place to factor.

$$(x+1)(x-3)=0 \qquad \text{Factor.}$$

We have a product that equals zero. Use the zero product property to complete the solution.

$$\begin{aligned} x+1&=0 &\quad\text{or}\quad&& x-3&=0 \\ x&=-1 &&& x&=3 \end{aligned}$$

Thus, the solutions of $x^2=2x+3$ are $x=-1$ and $x=3$.

Graphical solution. Load each side of the equation $x^2=2x+3$ into the **Y=** menu of your graphing calculator, $y=x^2$ in **Y1**, $y=2x+3$ in **Y2** (see Figure 1.10). Select **6:ZStandard** from the ZOOM menu to produce the image at the right in Figure 1.10.

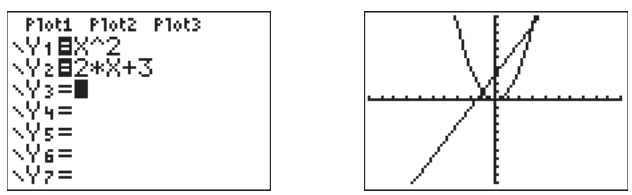


Figure 1.10: Sketch $y=x^2$ and $y=2x+3$.

One of the intersection points is visible on the left, but the second point of intersection is very near the top of the screen at the right (see Figure 1.10). Let's extend the top of the screen a bit. Press the WINDOW button and make adjustments to **Ymin** and **Ymax** (see Figure 1.11), then press the GRAPH button to adopt the changes.

Note that both points of intersection are now visible in the viewing window (see Figure 1.11). To find the coordinates of the points of intersection, select **5:intersect** from the CALC menu. Press the ENTER key to accept the "First curve," press ENTER again to accept the "Second curve," then press ENTER again to accept the current position of the cursor as your guess. The result is shown in the image on the left in Figure 1.12. Repeat the process to find the second point of intersection, only when it comes time to enter your "Guess," use the right-arrow key to move the cursor closer to the second point of intersection than the first.

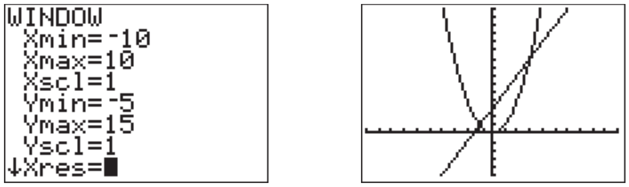


Figure 1.11: Adjusting the view.

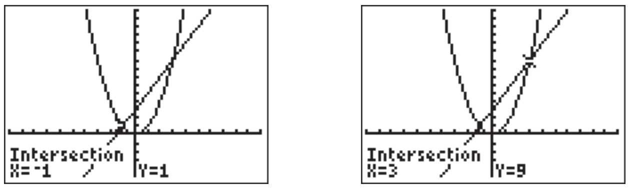
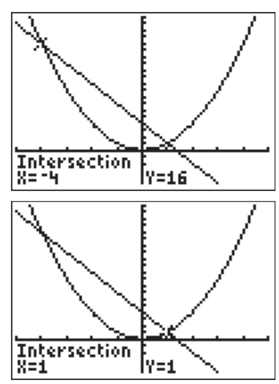


Figure 1.12: Use 5:intersect from the CALC menu to find points of intersection.

Reporting the solution on your homework: Duplicate the image in your calculator's viewing window on your homework page. Use a ruler to draw all lines, but freehand any curves.

- Label the horizontal and vertical axes with x and y, respectively (see Figure 1.13).
- Place your WINDOW parameters at the end of each axis (see Figure 1.13).
- Label each graph with its equation (see Figure 1.13).
- Drop dashed vertical lines through each point of intersection. Shade and label the x-values of the points where the dashed vertical line crosses the x-axis. These are the solutions of the equation $x^2=2x+3$ (See Figure 1.13).

Answer: -4, 1



Finally, note how the graphical solutions of $x^2 = 2x + 3$, namely $x = -1$ and $x = 3$, match the solutions found using the algebraic method. This is solid evidence that both methods of solution are correct. However, it doesn't hurt to check the final answers in the original equation, substituting -1 for x and 3 for x.

$$\begin{aligned} x^2 &= 2x+3 \\ (-1)^2 &= 2(-1)+3 \\ 1 &= -2+3 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x^2 &= 2x+3 \\ (3)^2 &= 2(3)+3 \\ 9 &= 6+3 \end{aligned}$$

Because the last two statements are true statements, the solutions $x = -1$ and $x = 3$ check in the original equation $x^2 = 2x + 3$.

□

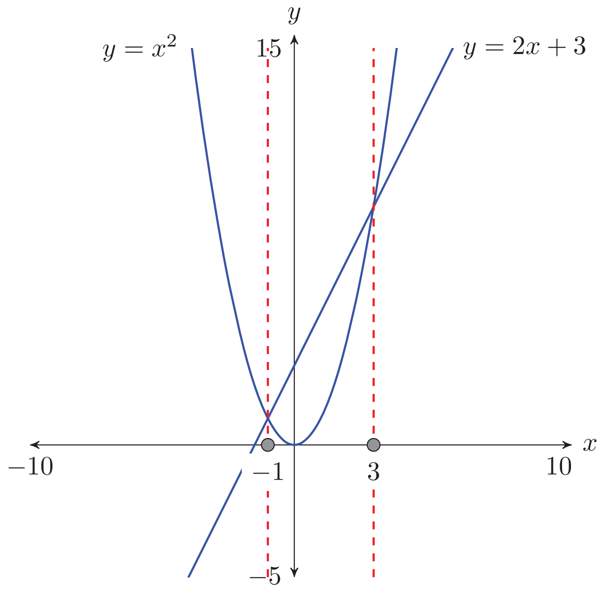


Figure 1.13: Reporting your graphical solution on your homework.

You Try It!

Solve the equation $x^2 - 21x + 90 = 0$ both algebraically and graphically, then compare your answers.

EXAMPLE 7. Solve the equation $x^2 - 4x - 96 = 0$ both algebraically and graphically, then compare your answers.

Solution: Because there is a power of x larger than one, the equation $x^2 - 4x - 96 = 0$ is nonlinear. We already have one side zero, so we can proceed with the factoring. Begin listing integer pairs whose product is $ac = -96$, mindful of the fact that we need a pair whose sum is $b = -4$.

$$\begin{array}{c} 1, -96 \\ 2, -48 \\ 3, -32 \\ 4, -24 \\ 6, -16 \\ \boxed{8, -12} \end{array}$$

Note that we stopped the listing process as soon as we encountered a pair whose sum was $b = -4$. "Drop" this pair in place to factor the trinomial.

$$x^2 - 4x - 96 = 0 \quad \text{Original equation.}$$

$$(x + 8)(x - 12) = 0 \quad \text{Factor.}$$

We have a product that equals zero. Use the zero product property to complete the solution.

$$x + 8 = 0 \quad \text{or} \quad x - 12 = 0$$
$$x = -8 \qquad\qquad x = 12$$

Thus, the solutions of $x^2 - 4x - 96 = 0$ are $x = -8$ and $x = 12$.

Graphical solution. Load the equation $y = x^2 - 4x - 96$ in **Y1** in the **Y=** menu of your graphing calculator (see Figure 1.14). Select **6:ZStandard** from the ZOOM menu to produce the image at the right in Figure 1.14.

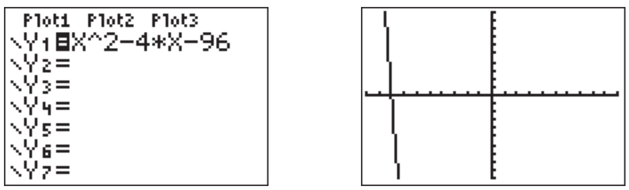


Figure 1.14: Sketch the graph of $y = x^2 - 4x - 96$.

When the degree of a polynomial is two, we're used to seeing some sort of parabola. In Figure 1.14, we saw the graph go down and off the screen, but we did not see it turn and come back up. Let's adjust the WINDOW parameters so that the vertex (turning point) of the parabola and both x-intercepts are visible in the viewing window. After some experimentation, the settings shown in Figure 1.15 reveal the vertex and the x-intercepts. Press the GRAPH button to produce the image at the right in Figure 1.15.

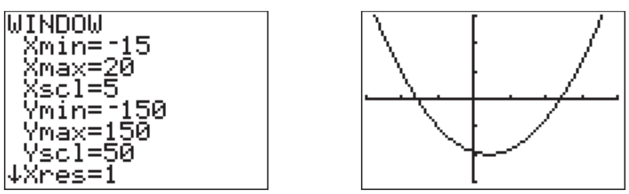


Figure 1.15: Adjusting the view.

Note that both x-intercepts of the parabola are now visible in the viewing window (see Figure 1.15). To find the coordinates of the x-intercepts, select **2:zero** from the CALC menu. Use the left- and right-arrow keys to move the cursor to the left of the first x-intercept, then press ENTER to mark the "Left bound." Next, move the cursor to the right of the first x-intercept, then press ENTER to mark the "Right bound." Press ENTER to accept the current position of the cursor as your "Guess." The result is shown in the image on the left in Figure 1.16. Repeat the process to find the coordinates of the second x-intercept. The result is show in the image on the right in Figure 1.16.

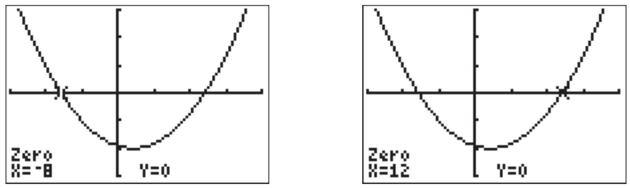


Figure 1.16: Use **2:zero** from the CALC menu to find the x-intercepts.

Reporting the solution on your homework: Duplicate the image in your calculator's viewing window on your homework page. Use a ruler to draw all lines, but freehand any curves.

- Label the horizontal and vertical axes with x and y, respectively (see Figure 1.17).
- Place your WINDOW parameters at the end of each axis (see Figure 1.17).
- Label the graph with its equation (see Figure 1.17).
- Drop dashed vertical lines through each x-intercept. Shade and label the x-values of the points where the dashed vertical line crosses the x-axis. These are the solutions of the equation $x^2 - 4x - 96 = 0$ (see Figure 1.17).

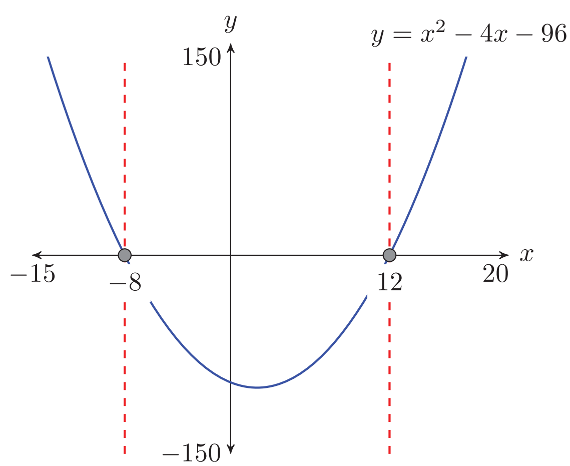


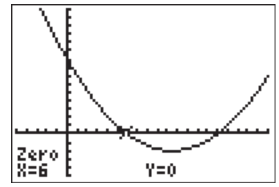
Figure 1.17: Reporting your graphical solution on your homework.

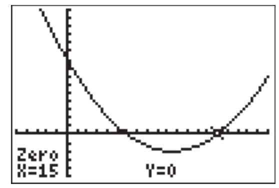
Finally, note how the graphical solutions of $x^2 - 4x - 96 = 0$, namely $x = -8$ and $x = 12$, match the solutions found using the algebraic method. This is solid evidence that both methods of solution are correct. However, it doesn't hurt to check the final answers in the original equation, substituting -8 for x and 12 for x.

$$\begin{aligned} x^2 - 4x - 96 &= 0 \\ (-8)^2 - 4(-8) - 96 &= 0 \\ 64 + 32 - 96 &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x^2 - 4x - 96 &= 0 \\ (12)^2 - 4(12) - 96 &= 0 \\ 144 - 48 - 96 &= 0 \end{aligned}$$

Because the last two statements are true statements, the solutions $x = -8$ and $x = 12$ check in the original equation $x^2 - 4x - 96 - 0$.

Answer: 6, 15





□

1.3 Exercises

In Exercises 1-6, compare the given trinomial with $ax^2 + bx + c$, then list ALL integer pairs whose product equals ac. Circle the pair whose sum equals b, then use this pair to help factor the given trinomial.

1. $x^2 + 7x - 18$

2. $x^2 + 18x + 80$

3. $x^2 - 10x + 9$

4. $x^2 + 12x + 27$

5. $x^2 + 14x + 45$

6. $x^2 + 9x + 20$

In Exercises 7-12, compare the given trinomial with ax^2+bx+c, then begin listing integer pairs whose product equals ac. Cease the list process when you discover a pair whose sum equals b, then circle and use this pair to help factor the given trinomial.

7. $x^2 - 16x + 39$

8. $x^2 - 16x + 48$

9. $x^2 - 26x + 69$

10. $x^2 - 22x + 57$

11. $x^2 - 25x + 84$

12. $x^2 + 13x - 30$

In Exercises 13-18, compare the given trinomial with $ax^2 + bx + c$, then compute ac. Try to mentally discover the integer pair whose product is ac and whose sum is b. Factor the trinomial by "dropping this pair in place." *Note: If you find you cannot identify the pair mentally, begin listing integer pairs whose product equals ac, then cease the listing process when you encounter the pair whose sum equals b.*

13. $x^2 - 13x + 36$

14. $x^2 + x - 12$

15. $x^2 + 10x + 21$

16. $x^2 - 17x + 66$

17. $x^2 - 4x - 5$

18. $x^2 - 20x + 99$

In Exercises 19-24, use an algebraic technique to solve the given equation.

19. $x^2 = -7x + 30$

20. $x^2 = -2x + 35$

21. $x^2 = -11x - 10$

22. $x^2 = x + 72$

23. $x^2 = -15x - 50$

24. $x^2 = -7x - 6$

In Exercises 25-30, use an algebraic technique to solve the given equation.

25. $60 = x^2 + 11x$

26. $-92 = x^2 - 27x$

27. $-11 = x^2 - 12x$

28. $80 = x^2 - 16x$

29. $56 = x^2 + 10x$

30. $66 = x^2 + 19x$

In Exercises 31-36, use an algebraic technique to solve the given equation.

31. $x^2 + 20 = -12x$

32. $x^2 - 12 = 11x$

33. $x^2 - 36 = 9x$

34. $x^2 + 6 = 5x$

35. $x^2 + 8 = -6x$

36. $x^2 + 77 = 18x$

In Exercises 37-40, perform each of the following tasks:

i) Use a strictly algebraic technique to solve the given equation.

ii) Use the **5:intersect** utility on your graphing calculator to solve the given equation. Report the results found using graphing calculator as shown in Example 6.

37. $x^2 = x + 12$

38. $x^2 = 20 - x$

39. $x^2 + 12 = 8x$

40. $x^2 + 7 = 8x$

In Exercises 41-44, perform each of the following tasks:

i) Use a strictly algebraic technique to solve the given equation.

ii) Use the **2:zero** utility on your graphing calculator to solve the given equation. Report the results found using graphing calculator as shown in Example 7.

41. $x^2 - 6x - 16 = 0$

42. $x^2 + 7x - 18 = 0$

43. $x^2 + 10x - 24 = 0$

44. $x^2 - 9x - 36 = 0$

1.3 Answers

1. $(x-2)(x+9)$

3. $(x-1)(x-9)$

5. $(x+5)(x+9)$

7. $(x-3)(x-13)$

9. $(x-3)(x-23)$

11. $(x-4)(x-21)$

13. $(x-4)(x-9)$

15. $(x+3)(x+7)$

17. $(x+1)(x-5)$

19. $x=3,-10$

21. $x=-1,-10$

23. $x=-5,-10$

25. $x=4,-15$

27. $x=1,11$

29. $x=4,-14$

31. $x=-2,-10$

33. $x=-3,12$

35. $x=-2,-4$

37. $x=-3,4$

39. $x=2,6$

41. $x=8,-2$

43. $x=-12,2$

1.4 Factoring $ax^2 + bx + c$ when $a \neq 1$

In this section we continue to factor trinomials of the form $ax^2 + bx + c$. In the last section, all of our examples had $a = 1$, and we were able to "Drop in place" our circled integer pair. However, in this section, $a \neq 1$, and we'll soon see that we will not be able to use the "Drop in place" technique. However, readers will be pleased to learn that the ac-method will still apply.

You Try It!

Factor: $3x^2 + 13x + 14$

EXAMPLE 1. Factor: $2x^2 - 7x - 15$.

Solution: We proceed as follows:

1. Compare $2x^2 - 7x - 15$ with $ax^2 + bx + c$ and identify $a = 2$, $b = -7$, and $c = -15$. Note that the leading coefficient is $a = 2$, so this case is different from all of the cases discussed in Section 6.3.

2. Calculate ac. Note that $ac = (2)(-15)$, so $ac = -30$.

3. List all integer pairs whose product is $ac = -30$.

$1, -30$	$-1, 30$
$2, -15$	$-2, 15$
$3, -10$	$-3, 10$
$5, -6$	$-5, 6$

4. Circle the ordered pair whose sum is $b = -7$.

$1, -30$	$-1, 30$
$2, -15$	$-2, 15$
$\boxed{3, -10}$	$-3, 10$
$5, -6$	$-5, 6$

5. Note that if we "drop in place" our circled ordered pair, $(x+3)(x-10) \neq 2x^2 - 7x - 15$. Right off the bat, the product of the terms in the "First" position does not equal $2x^2$. Instead, we break up the middle term of $2x^2 - 7x - 15$ into a sum of like terms using our circled pair of integers 3 and -10.

$$2x^2 - 7x - 15 = 2x^2 + 3x - 10x - 15$$

Now we factor by grouping. Factor x out of the first two terms and -5 out of the second two terms.

$$= x\,(2x + 3) - 5\,(2x + 3)$$

Now we can factor out $(2x + 3)$.

$$= (x - 5)(2x + 3)$$

6. Use the FOIL shortcut to mentally check your answer. To multiply $(x-5)(2x+3)$, use these steps:
   - Multiply the terms in the "First" positions: $2x^2$.
   - Multiply the terms in the "Outer" and "Inner" positions and combine the results mentally: $3x - 10x = -7x$.
   - Multiply the terms in the "Last" positions: -15.

   That is:

   $$(x-5)(2x+3) = \overset{F}{2x^2} + \overset{O}{3x} - \overset{I}{10x} - \overset{L}{15}$$

   Combining like terms, $(x-5)(2x+3) = 2x^2 - 7x - 15$, which is the original trinomial, so our solution checks. Note that if you combine the "Outer" and "Inner" products mentally, the check goes even faster.

Answer: $(x+2)(3x+7)$

□

Speeding Things Up a Bit

Some readers might already be asking "Do I really have to list all of those ordered pairs if I already see the pair I need?" The answer is "No!" If you see the pair you need, use it to break up the middle term of the trinomial as a sum of like terms.

You Try It!

Factor: $2x^2 - 9x + 10$

EXAMPLE 2. Factor: $3x^2 - 7x - 6$.

Solution: Compare $3x^2-7x-6$ with ax^2+bx+c and note that $a=3$, $b=-7$, and $c=-6$. Calculate $ac=(3)(-6)$, so $ac=-18$. Now can you think of an integer pair whose product is $ac=-18$ and whose sum is $b=-7$? For some, the pair just pops into their head: 2 and -9. Break up the middle term into a sum of like terms using the pair 2 and -9.

$$\begin{aligned} &3x^2 - 7x - 6 \\ &\quad = 3x^2 + 2x - 9x - 6 && -7x = 2x - 9x. \\ &\quad = x(3x+2) - 3(3x+2) && \text{Factor by grouping.} \\ &\quad = (x-3)(3x+2) && \text{Factor out } (3x+2). \end{aligned}$$

Use the FOIL shortcut to check your answer.

$$(x-3)(3x+2) = \overset{F}{3x^2} + \overset{O}{2x} - \overset{I}{9x} - \overset{L}{6}$$

Combining like terms, $(x-3)(3x+2) = 3x^2-7x-6$, the original trinomial. Our solution checks. Note that if you combine the "Outer" and "Inner" products mentally, the check goes even faster.

Answer: $(x-2)(2x-5)$

On the other hand, some readers might be saying "Well, the needed ordered pair is not popping into my head. Do I have a way of cutting down the work?" The answer is "Yes!" As you are listing the ordered pairs whose product equals ac, be mindful that you need the ordered pair whose sum is b. If you stumble across the needed pair, stop the listing process and "drop" your ordered pair in place.

You Try It!

EXAMPLE 3. Factor: $3x^2 - 33x + 54$.

Factor: $5x^2 - 35x - 40$

Solution: Compare $3x^2 - 33x + 54$ with $ax^2 + bx + c$ and note that $a = 3$, $b = -33$, and $c = 54$. Calculate $ac = (3)(54)$, so $ac = 162$. Ouch! That's a big number! However, start listing the integer pairs whose product is $ac = 162$, but be mindful that you need an integer pair whose sum is $b = -33$.

$$\begin{array}{ll} 1, 162 & \\ 2, 81 & \\ 3, 54 & \\ 6, 27 & \boxed{-6, -27} \end{array}$$

As soon as we wrote down the pair 6 and 27, our mind said "the sum of 6 and 7 is 33." However, we need the sum to equal $b = -33$, so we boxed -6 and -27 instead. Next, we break up the middle term into a sum of like terms using our circled pair.

$$\begin{aligned} &3x^2 - 33x - 54 \\ &\quad = 3x^2 - 6x - 27x - 54 && -33x = -6x - 27x. \\ &\quad = 3x\,(x-2) - 27(x-2) && \text{Factor by grouping.} \\ &\quad = (3x-27)(x-2) && \text{Factor out } (x-2). \end{aligned}$$

Oh-oh! Now we realize we can factor 3 out of each term in the first factor!

$$= 3(x-9)(x-2)$$

We missed taking out the GCF! Let's try again, only this time let's do what we are always supposed to do in the first step: Factor out the GCF.

$$3x^2 - 33x + 54 = 3(x^2 - 11x + 18)$$

Comparing $x^2 - 11x + 18$ with $ax^2 + bx + c$, we see that $a = 1$, $b = -11$, and $c = 18$. We need an integer pair whose product is $ac = 18$ and whose sum is $b = -11$. Note that these numbers are considerably smaller than the numbers we had to deal with when we forgot to first factor out the GCF. Because the numbers are smaller, the integer pair -9 and -2 easily comes to mind. Furthermore, because $a = 1$, we can factor $x^2 - 11x + 18$ by simply dropping the integer pair -9 and -2 in place.

$$3(x^2 - 11x + 18) = 3(x - 9)(x - 2)$$

Answer: $5(x - 8)(x + 1)$

A far simpler solution! □

In Example 3, we saw how much more difficult we made the problem by forgetting to first factor out the greatest common factor (GCF). Let's try not to make that mistake again.

First rule of factoring. The first step in factoring any polynomial is to factor out the greatest common factor.

You Try It!

Factor: $12x^4 + 2x^3 - 30x^2$

EXAMPLE 4. Factor: $30x^3 - 21x^2 - 18x$.

Solution: Note that the GCF of $30x^3$, $21x^2$, and $18x$ is $3x$. Factor out this GCF.

$$\begin{aligned} 30x^3 - 21x^2 - 18x &= 3x \cdot 10x^2 - 3x \cdot 7x - 3x \cdot 6 \\ &= 3x(10x^2 - 7x - 6) \end{aligned}$$

Next, compare $10x^2 - 7x - 6$ with $ax^2 + bx + c$ and note that $a = 10$, $b = -7$, and $c = -6$. Start listing the integer pairs whose product is $ac = -60$, but be mindful that you need an integer pair whose sum is $b = -7$.

$$\begin{gathered} 1, -60 \\ 2, -30 \\ 3, -20 \\ 4, -15 \\ \boxed{5, -12} \end{gathered}$$

Break up the middle term into a sum of like terms using our circled pair.

$$\begin{aligned} &3x(10x^2 - 7x - 6) \\ &\quad = 3x(10x^2 + 5x - 12x - 18) && -7x = 5x - 12x. \\ &\quad = 3x\left[5x(2x + 1) - 6(2x + 1)\right] && \text{Factor by grouping.} \\ &\quad = 3x(5x - 6)(2x + 1) && \text{Factor out a } 2x + 1. \end{aligned}$$

Hence, $30x^3 - 21x^2 - 18x = 3x(5x - 6)(2x + 1)$.

Check: First, use the FOIL shortcut to multiply the two binomial factors, then distribute the monomial factor.

$$\begin{aligned} 3x(5x-6)(2x+1) &= 3x(10x^2-7x-6) && \text{Apply the FOIL shortcut.} \\ &= 30x^3-21x^2-18x && \text{Distribute the } 3x. \end{aligned}$$

Because this is the original polynomial, the solution checks.

Answer: $2x^2(3x+5)(2x-3)$

Nonlinear Equations Revisited

Let's use the factoring technique of this chapter to solve some nonlinear equations.

You Try It!

Solve the equation $5x^2 = 12x + 9$ both algebraically and graphically, then compare your answers.

EXAMPLE 5. Solve the equation $2x^2 = 13x - 20$ both algebraically and graphically, then compare your answers.

Solution: Because there is a power of x larger than one, the equation is nonlinear. Make one side equal to zero.

$$\begin{aligned} 2x^2 &= 13x-20 && \text{Original equation.} \\ 2x^2-13x+20 &= 0 && \text{Make one side zero.} \end{aligned}$$

Compare $2x^2 - 13x + 20$ with $ax^2 + bx + c$ and note that $a = 2$, $b = -13$ and $c = 20$. We need an integer pair whose product is $ac = 40$ and whose sum is $b = -13$. The integer pair -5 and -8 comes to mind. Write the middle term as a sum of like terms using this pair.

$$\begin{aligned} 2x^2-5x-8x+20 &= 0 && -13x = -5x-8x. \\ x(2x-5)-4(2x-5) &= 0 && \text{Factor by grouping.} \\ (x-4)(2x-5) &= 0 && \text{Factor out } 2x-5. \end{aligned}$$

We have a product that equals zero. Use the zero product property to complete the solution.

$$\begin{aligned} x-4 &= 0 \qquad \text{or} \qquad & 2x-5 &= 0 \\ x &= 4 & 2x &= 5 \\ & & x &= \frac{5}{2} \end{aligned}$$

Thus, the solutions of $2x^2 = 13x - 20$ are $x = 4$ and $x = 5/2$.

Graphical solution. Load each side of the equation $2x^2 = 13x - 20$ into the **Y=** menu of your graphing calculator, $y = 2x^2$ in **Y1**, $y = 13x - 20$ in **Y2**

(see Figure 1.18). Select **6:ZStandard** from the ZOOM menu to produce the image on the left in Figure 1.18. However, even after adjusting the WINDOW parameters (**Xmin** $= -10$, **Xmax** $= 10$, **Ymin** $= -10$, and **Ymax** $= 60$), the image resulting from pushing the GRAPH button (see the image on the right in Figure 1.18) does not clearly show the two points of intersection.

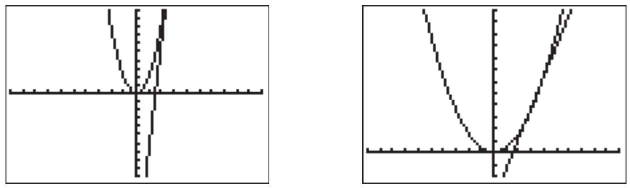


Figure 1.18: Sketch $y = 2x^2$ and $y = 13x - 20$.

Let's switch our strategy and work with the equation

$$2x^2 - 13x + 20 = 0$$

instead. Load $y = 2x^2 - 13x + 20$ into **Y1** in the Y= menu, then select **6:ZStandard** from the ZOOM menu to produce the image at the right in Figure 1.19.

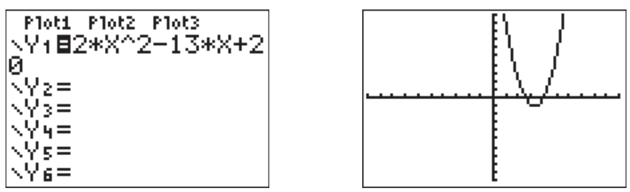


Figure 1.19: Sketch $y = 2x^2 - 13x + 20$.

To find the solutions of $2x^2 - 13x + 20 = 0$, we must identify the x-intercepts of the graph in Figure 1.18. Select **2:zero** from the CALC menu, then move the left- and right=arrows to move the cursor to the left of the first x-intercept. Press ENTER to mark the "Left bound," then move the cursor to the right of the x-intercept and press ENTER to mark the "Right bound." Finally, press ENTER to use the current position of the cursor for your "Guess." The result is shown in the image on the left in Figure 1.20. Repeat the process to find the rightmost x-intercept. The result is shown in the image on the right in Figure 1.20.

Reporting the solution on your homework: Duplicate the image in your calculator's viewing window on your homework page. Use a ruler to draw all lines, but freehand any curves.

- Label the horizontal and vertical axes with x and y, respectively (see Figure 1.21).

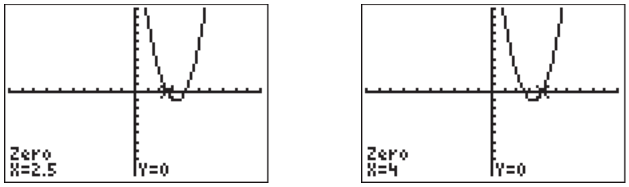


Figure 1.20: Use 2:zero from the CALC menu to find the x-intercepts.

- Place your WINDOW parameters at the end of each axis (see Figure 1.21).
- Label the graph with its equation (see Figure 1.21).
- Drop dashed vertical lines through each x-intercept. Shade and label the x-values of the points where the dashed vertical line crosses the x-axis. These are the solutions of the equation $2x^2 - 13x + 20 = 0$ (see Figure 1.21).

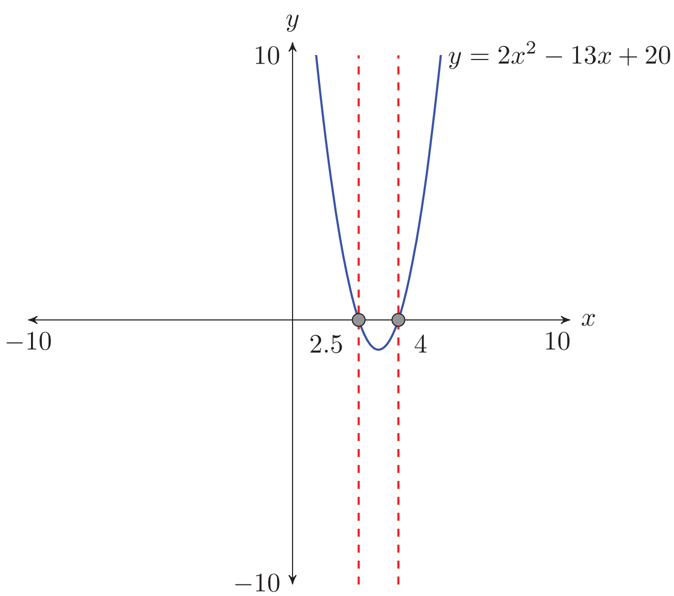
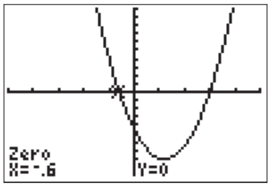
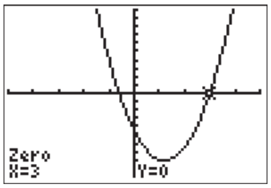


Figure 1.21: Reporting your graphical solution on your homework.

Finally, note how the graphical solutions of $2x^2 - 13x + 20 = 0$, namely $x = 2.5$ and $x = 4$, match the solutions $x = 5/2$ and $x = 4$ found using the algebraic method. This is solid evidence that both methods of solution are correct. However, it doesn't hurt to check the final answers in the original equation, Answer: $-3/5$, 3





substituting $5/2$ for x and 4 for x.

$$\begin{aligned} 2x^2 &= 13x - 20 \\ 2\left(\frac{5}{2}\right)^2 &= 13\left(\frac{5}{2}\right) - 20 \\ 2\left(\frac{25}{4}\right) &= 13\left(\frac{5}{2}\right) - 20 \\ \frac{25}{2} &= \frac{65}{2} - \frac{40}{2} \end{aligned} \qquad \text{and} \qquad \begin{aligned} 2x^2 &= 13x - 20 \\ 2(4)^2 &= 13(4) - 20 \\ 2(16) &= 13(4) - 20 \\ 32 &= 52 - 20 \end{aligned}$$

Because the last two statements are true statements, the solutions $x = 5/2$ and $x = 4$ check in the original equation $2x^2 = 13x - 20$.

□

You Try It!

Solve the equation $4x^3 = -x^2 + 14x$ both algebraically and graphically, then compare your answers.

EXAMPLE 6. Solve the equation $2x^3 + x^2 = 28x$ both algebraically and graphically, then compare your answers.

Solution: Because there is a power of x larger than one, the equation is nonlinear. Make one side equal to zero.

$$\begin{aligned} 2x^3 + x^2 &= 28x && \text{Original equation.} \\ 2x^3 + x^2 - 28x &= 0 && \text{Make one side zero.} \end{aligned}$$

Note that the GCF of $2x^3$, x^2, and $28x$ is x. Factor out x.

$$x(2x^2 + x - 28) = 0 \qquad \text{Factor out the GCF.}$$

Compare $2x^2+x-28$ with ax^2+bx+c and note that $a = 2$, $b = 1$ and $c = -28$. We need an integer pair whose product is $ac = -56$ and whose sum is $b = 1$. The integer pair -7 and 8 comes to mind. Write the middle term as a sum of like terms using this pair.

$$\begin{aligned} x(2x^2 - 7x + 8x - 28) &= 0 && x = -7x + 8x. \\ x\big[x(2x - 7) + 4(2x - 7)\big] &= 0 && \text{Factor by grouping.} \\ x(x + 4)(2x - 7) &= 0 && \text{Factor out } 2x - 7. \end{aligned}$$

We have a product of three factors that equals zero. By the zero product property, at least one of the factors must equal zero.

$$x = 0 \quad \text{or} \quad x + 4 = 0 \quad \text{or} \quad 2x - 7 = 0$$
$$x = -4 \qquad\qquad 2x = 7$$
$$x = \frac{7}{2}$$

Thus, the solutions of $2x^3 + x^2 = 28x$ are $x = 0$, $x = -4$, and $x = 7/2$.

Graphical solution. Rather than working with $2x^3+x^2 = 28x$, graphing each side separately and finding where the graphs intersect, we will work instead with $2x^3 + x^2 - 28x = 0$, locating where the graph of $y = 2x^3 + x^2 - 28x$ crosses the x-axis. Load $y = 2x^3 + x^2 - 28x$ into **Y1** in the Y= menu, then select **6:ZStandard** from the ZOOM menu to produce the image at the right in Figure 1.22.

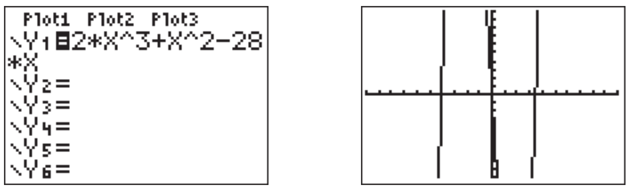


Figure 1.22: Sketch $y = 2x^3 + x^2 - 28x$.

In the image at the right in Figure 1.22, we saw the graph rise from the bottom of the screen, leave the top of the screen, return and leave via the bottom of the screen, and then finally return and leave via the top of the screen. Clearly, there are at least two turning points to the graph that are not visible in the current viewing window. Set the WINDOW settings as shown in the image on the left in Figure 1.23, then push the GRAPH button to produce the image on the right in Figure 1.23. Note that this window now shows the x-intercepts as well as the turning points of the graph of the polynomial.

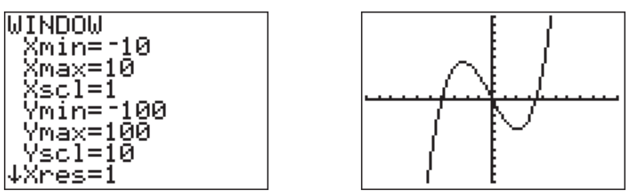


Figure 1.23: Adjusting the view so that the turning points of the polynomial are visible.

To find the solutions of $2x^3 + x^2 - 28x = 0$, we must identify the x-intercepts of the graph in Figure 1.23. Select **2:zero** from the CALC menu, then use the left- and right-arrow keys to move the cursor to the left of the first x-intercept. Press ENTER to mark the "Left bound," then move the cursor to the right of the x-intercept and press ENTER to mark the "Right bound." Finally, press ENTER to use the current position of the cursor for your "Guess." The result is shown in the first image on the left in Figure 1.24. Repeat the process to find the remaining x-intercepts. The results are shown in the next two images in Figure 1.24.

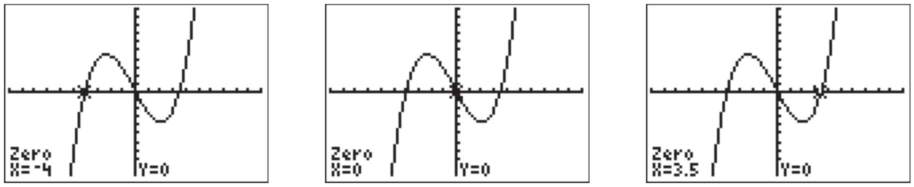
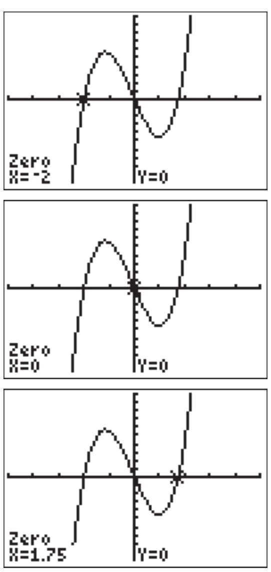


Figure 1.24: Use 2:zero from the CALC menu to find the x-intercepts.

Reporting the solution on your homework: Duplicate the image in your calculator's viewing window on your homework page. Use a ruler to draw all lines, but freehand any curves.

- Label the horizontal and vertical axes with x and y, respectively (see Figure 1.25).
- Place your WINDOW parameters at the end of each axis (see Figure 1.25).
- Label the graph with its equation (see Figure 1.25).
- Drop dashed vertical lines through each x-intercept. Shade and label the x-values of the points where the dashed vertical line crosses the x-axis. These are the solutions of the equation $2x^3+x^2-28x=0$ (see Figure 1.25).

Answer: -2, 0, $7/4$



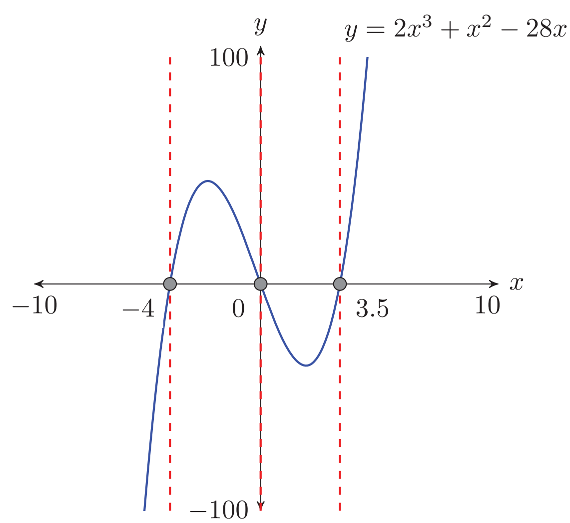


Figure 1.25: Reporting your graphical solution on your homework.

Finally, note how the graphical solutions of $2x^3+x^2-28x = 0$, namely $x = -4$, $x = 0$, and $x = 3.5$, match the solutions $x = -4$, $x = 0$, and $x = 7/2$ found using the algebraic method.

□

1.4 Exercises

In Exercises 1-6, compare the given trinomial with $ax^2 + bx + c$, then list ALL integer pairs whose product equals ac. Circle the pair whose sum equals b, then use this pair to help factor the given trinomial.

1. $6x^2 + 13x - 5$

2. $3x^2 - 19x + 20$

3. $4x^2 - x - 3$

4. $6x^2 - 23x + 7$

5. $3x^2 + 19x + 28$

6. $2x^2 - 9x - 18$

In Exercises 7-12, compare the given trinomial with $ax^2 + bx + c$, then begin listing integer pairs whose product equals ac. Cease the list process when you discover a pair whose sum equals b, then circle and use this pair to help factor the given trinomial.

7. $12x^2 - 23x + 5$

8. $8x^2 + 22x + 9$

9. $6x^2 + 17x + 7$

10. $4x^2 + 19x + 21$

11. $3x^2 + 4x - 32$

12. $4x^2 + x - 14$

In Exercises 13-18, compare the given trinomial with $ax^2 + bx + c$, then compute ac. Try to mentally discover the integer pair whose product is ac and whose sum is b. Use this pair to help factor the given trinomial. *Note: If you find you cannot identify the pair mentally, begin listing integer pairs whose product equals ac, then cease the listing process when you encounter the pair whose sum equals b.*

13. $3x^2 + 28x + 9$

14. $6x^2 + x - 1$

15. $4x^2 - 21x + 5$

16. $4x^2 - x - 14$

17. $6x^2 - 11x - 7$

18. $2x^2 - 17x + 21$

In Exercises 19-26, factor the trinomial.

19. $16x^5 - 36x^4 + 14x^3$

20. $12x^4 - 20x^3 + 8x^2$

21. $36x^4 - 75x^3 + 21x^2$

22. $6x^4 - 10x^3 - 24x^2$

23. $6x^4 - 33x^3 + 42x^2$

24. $15x^3 - 10x^2 - 105x$

25. $16x^4 - 36x^3 - 36x^2$

26. $40x^4 - 10x^3 - 5x^2$

In Exercises 27-38, use an algebraic technique to solve the given equation.

27. $4x^2 = -x + 18$

28. $2x^2 = 7x - 3$

29. $3x^2 + 16 = -14x$

30. $2x^2 - 20 = -3x$

31. $3x^2 + 30 = 23x$

32. $6x^2 - 7 = -11x$

33. $-7x - 3 = -6x^2$

34. $13x - 45 = -2x^2$

35. $26x - 9 = -3x^2$

36. $-23x + 7 = -6x^2$

37. $6x^2 = -25x + 9$

38. $2x^2 = 13x + 45$

In Exercises 39-42, perform each of the following tasks:

i) Use a strictly algebraic technique to solve the given equation.

ii) Use the **2:zero** utility on your graphing calculator to solve the given equation. Report the results found using graphing calculator as shown in Example 5.

39. $2x^2 - 9x - 5 = 0$

40. $2x^2 + x - 28 = 0$

41. $4x^2 - 17x - 15 = 0$

42. $3x^2 + 14x - 24 = 0$

In Exercises 43-46, perform each of the following tasks:

i) Use a strictly algebraic technique to solve the given equation.

ii) Use the **2:zero** utility on your graphing calculator to solve the given equation. Report the results found using graphing calculator as shown in Example 6.

43. $2x^3 = 3x^2 + 20x$

44. $2x^3 = 3x^2 + 35x$

45. $10x^3 + 34x^2 = 24x$

46. $6x^3 + 3x^2 = 63x$

1.4 Answers

1. $(2x+5)(3x-1)$

3. $(x-1)(4x+3)$

5. $(x+4)(3x+7)$

7. $(3x-5)(4x-1)$

9. $(2x+1)(3x+7)$

11. $(x+4)(3x-8)$

13. $(3x+1)(x+9)$

15. $(x-5)(4x-1)$

17. $(3x-7)(2x+1)$

19. $2x^3(2x-1)(4x-7)$

21. $3x^2(3x-1)(4x-7)$

23. $3x^2(x-2)(2x-7)$

25. $4x^2(x-3)(4x+3)$

27. $x = 2, -\frac{9}{4}$

29. $x = -2, -\frac{8}{3}$

31. $x = 6, \frac{5}{3}$

33. $x = -\frac{1}{3}, \frac{3}{2}$

35. $x = -9, \frac{1}{3}$

37. $x = \frac{1}{3}, -\frac{9}{2}$

39. $x = -1/2, 5$

41. $x = -3/4, 5$

43. $x = 0, -5/2, 4$

45. $x = 0, -4, 3/5$

1.5 Factoring Special Forms

In this section we revisit two special product forms we learned in Beginning Algebra, the first of which is *squaring a binomial.*

Squaring a binomial. Here are two earlier rules for squaring a binomial.

1. $(a+b)^2 = a^2 + 2ab + b^2$
2. $(a-b)^2 = a^2 - 2ab + b^2$

Perfect Square Trinomials

To square a binomial such as $(a+b)^2$, proceed as follows:

1. Square the first term: a^2
2. Multiply the first and second term, then double: $2ab$
3. Square the last term: b^2

You Try It!

Expand: $(5a+2b)^2$

EXAMPLE 1. Expand: $(2x+3y)^2$

Solution: Using the pattern $(a+b)^2 = a^2+2ab+b^2$, we can expand $(2x+3y)^2$ as follows:

$$\begin{aligned}(2x+3y)^2 &= (2x)^2 + 2(2x)(3y) + (3y)^2 \\ &= 4x^2 + 6xy + 9y^2\end{aligned}$$

Note how we square the first and second terms, then produce the middle term of our answer by multiplying the first and second terms and doubling.

Answer: $25a^2 + 20ab + 4b^2$

□

You Try It!

Expand: $(2s^3 - 7t)^2$

EXAMPLE 2. Expand: $(3u^2 - 5v^2)^2$

Solution: Using the pattern $(a-b)^2 = a^2-2ab+b^2$, we can expand $(3u^2-5v^2)^2$ as follows:

$$\begin{aligned}(3u^2-5v^2)^2 &= (3u^2)^2 - 2(3u^2)(5v^2) + (5v^2)^2 \\ &= 9u^4 - 30u^2v^2 + 25v^4\end{aligned}$$

Note that the sign of the middle term is negative this time. The first and last terms are still positive because we are squaring.

Answer: $4s^6 - 28s^3t + 49t^2$

□

Once you've squared a few binomials, it's time to do all of the work in your head. (i) Square the first term; (ii) multiply the first and second term and double the result; and (iii) square the second term.

You Try It!

Expand: $(5x^4 - 3)^2$

EXAMPLE 3. Expand each of the following:

a) $(2y - 3)^2$ b) $(4a - 3b)^2$ c) $(x^3 + 5)^2$

Solution: Using the pattern $(a \pm b)^2 = a^2 \pm 2ab + b^2$, we expand each binomially mentally, writing down the answer without any intermediate steps.

a) $(2y - 3)^2 = 4y^2 - 12y + 9$

b) $(4a - 3b)^2 = 16a^2 - 24ab + 9b^2$

c) $(x^3 + 5)^2 = x^6 + 10x^3 + 25$

Answer: $25x^8 - 30x^4 + 9$

□

Now, because factoring is "unmultiplying," it should be a simple matter to reverse the process of Example 3.

You Try It!

Factor: $25x^8 - 30x^4 + 9$

EXAMPLE 4. Factor each of the following trinomials:

a) $4y^2 - 12y + 9$ b) $16a^2 - 24ab + 9b^2$ c) $x^6 + 10x^3 + 25$

Solution: Because of the work already done in Example 3, it is a simple task to factor each of these trinomials.

a) $4y^2 - 12y + 9 = (2y - 3)^2$

b) $16a^2 - 24ab + 9b^2 = (4a - 3b)^2$

c) $x^6 + 10x^3 + 25 = (x^3 + 5)^2$

Answer: $(5x^4 - 3)^2$

□

Each of the trinomials in Example 4 is an example of a *perfect square trinomial.*

Perfect square trinomial. If a trinomial $a^2 + 2ab + b^2$ is the square of a binomial, as in $(a+b)^2$, then the trinomial is called a *perfect square trinomial.*

So, how does one recognize a perfect square trinomial? If the first and last terms of a trinomial are perfect squares, then you should suspect that you may be dealing with a perfect square trinomial. However, you also have to have the correct middle term in order to have a perfect square trinomial.

You Try It!

Factor: $16x^2 + 72x + 81$

List of Squares

n	n^2
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100
11	121
12	144
13	169
14	196
15	225
16	256
17	289
18	324
19	361
20	400
21	441
22	484
23	529
24	576
25	625

EXAMPLE 5. Factor each of the following trinomials:

a) $9x^2 - 42x + 49$ b) $49a^2 + 70ab + 25b^2$ c) $4x^2 - 37x + 9$

Solution: Note that the first and last terms of each trinomial are perfect squares.

a) In the trinomial $9x^2 - 42x + 49$, note that $(3x)^2 = 9x^2$ and $7^2 = 49$. Hence, the first and last terms are perfect squares. Taking the square roots, we suspect that $9x^2 - 42x + 49$ factors as follows:

$$9x^2 - 42x + 49 \stackrel{?}{=} (3x - 7)^2$$

However, we must check to see if the middle term is correct. Multiply $3x$ and 7, then double: $2(3x)(7) = 42x$. Thus, the middle term is correct and therefore

$$9x^2 - 42x + 49 = (3x - 7)^2.$$

b) In the trinomial $49a^2+70ab+25b^2$, note that $(7a)^2 = 49a^2$ and $(5b)^2 = 25b^2$. Hence, the first and last terms are perfect squares. Taking the square roots, we suspect that $49a^2 + 70ab + 25b^2$ factors as follows:

$$49a^2 + 70ab + 25b^2 \stackrel{?}{=} (7a + 5b)^2$$

However, we must check to see if the middle term is correct. Multiply $7a$ and $5b$, then double: $2(7a)(5b) = 70ab$. Thus, the middle term is correct and therefore

$$49a^2 + 70ab + 25b^2 = (7a + 5b)^2.$$

c) In the trinomial $4x^2 - 37x + 9$, note that $(2x)^2 = 4x^2$ and $(3)^2 = 9$. Hence, the first and last terms are perfect squares. Taking the square roots, we suspect that $4x^2 - 37x + 9$ factors as follows:

$$4x^2 - 37x + 9 \stackrel{?}{=} (2x - 3)^2$$

However, we must check to see if the middle term is correct. Multiply $2x$ and 3, then double: $2(2x)(3) = 12x$. However, this is not the middle term of $4x^2 - 37x + 9$, so this factorization is incorrect! We must find another way to factor this trinomial.

Comparing $4x^2 - 37x + 9$ with $ax^2 + bx + c$, we need a pair of integers whose product is $ac = 36$ and whose sum is $b = -37$. The integer pair -1 and -36 comes to mind. Replace the middle term as a sum of like terms using this ordered pair.

$$\begin{aligned} 4x^2 - 37x + 9 &= 4x^2 - x - 36x + 9 && -37x = -x - 36x. \\ &= x(4x - 1) - 9(4x - 1) && \text{Factor by grouping.} \\ &= (x - 9)(4x - 1) && \text{Factor out } 4x - 1. \end{aligned}$$

This example clearly demonstrates how important it is to check the middle term.

Answer: $(4x + 9)^2$

□

Remember the first rule of factoring!

The first rule of factoring. The first step to perform in any factoring problem is factor out the GCF.

You Try It!

Factor: $-4x^3 - 24x^2 - 36x$

EXAMPLE 6. Factor each of the following trinomials:

a) $2x^3y + 12x^2y^2 + 18xy^3$ b) $-4x^5 + 32x^4 - 64x^3$

Solution: Remember, first factor out the GCF.

a) In the trinomial $2x^3y + 12x^2y^2 + 18xy^3$, we note that the GCF of $2x^3y$, $12x^2y^2$, and $18xy^3$ is $2xy$. We first factor out $2xy$.

$$2x^3y + 12x^2y^2 + 18xy^3 = 2xy(x^2 + 6xy + 9y^2)$$

We now note that the first and last terms of the resulting trinomial factor are perfect squares, so we take their square roots and factors as follows.

$$= 2xy(x + 3y)^2$$

Of course, the last factorization is correct only if the middle term is correct. Because $2(x)(3y) = 6xy$ matches the middle term of $x^2 + 6xy + 9y^2$, we do have a perfect square trinomial and our result is correct.

b) In the trinomial $-4x^5 + 32x^4 - 64x^3$, we note that the GCF of $4x^5$, $32x^4$, and $64x^3$ is $4x^3$. We first factor out $4x^3$.

$$-4x^5 + 32x^4 - 64x^3 = 4x^3(-x^2 + 8x - 16)$$

However, the first and third terms of $-x^2 + 8x - 16$ are negative, and thus are not perfect squares. Let's begin again, this time factoring out $-4x^3$.

$$-4x^5 + 32x^4 - 64x^3 = -4x^3(x^2 - 8x + 16)$$

This time the first and third terms of $x^2 - 8x + 16$ are perfect squares. We take their square roots and write:

$$= -4x^3(x - 4)^2$$

Again, this last factorization is correct only if the middle term is correct. Because $2(x)(4) = 8x$, we do have a perfect square trinomial and our result is correct.

Answer: $-4x(x + 3)^2$

□

The Difference of Squares

The second special product form we learned in Intermediate Algebra was the *difference of squares.*

The difference of squares. Here is the difference of squares rule.

$$(a + b)(a - b) = a^2 - b^2$$

If you are multiplying two binomials which have the exact same terms in the "First" positions and the exact same terms in the "Last" positions, but one set is separated by a plus sign while the other set is separated by a minus sign, then multiply as follows:

1. Square the first term: a^2
2. Square the second term: b^2
3. Place a minus sign between the two squares.

You Try It!

Expand: $(4x - 3y)(4x + 3y)$

EXAMPLE 7. Expand each of the following:

a) $(3x + 5)(3x - 5)$ b) $(a^3 - 2b^3)(a^3 + 2b^3)$

Solution: We apply the difference of squares pattern to expand each of the given problems.

a) In $(3x + 5)(3x - 5)$, we have the exact same terms in the "First" and "Last" positions, with the first set separated by a plus sign and the second set separated by a minus sign.

   a) Square the first term: $(3x)^2 = 9x^2$
   b) Square the second term: $5^2 = 25$
   c) Place a minus sign between the two squares.

   Hence:

$$(3x + 5)(3x - 5) = 9x^2 - 25$$

b) In $(a^3 - 2b^3)(a^3 + 2b^3)$, we have the exact same terms in the "First" and "Last" positions, with the first set separated by a minus sign and the second set separated by a plus sign.

   a) Square the first term: $(a^3)^2 = a^6$
   b) Square the second term: $(2b^3)^2 = 4b^6$
   c) Place a minus sign between the two squares.

   Hence:

$$(a^3 - 2b^3)(a^3 + 2b^3) = a^6 - 4b^6$$

Answer: $16x^2 - 9y^2$

□

Because factoring is "unmultiplying," is should be a simple matter to reverse the process of Example 7.

You Try It!

Factor: $81x^2 - 49$

EXAMPLE 8. Factor each of the following:

a) $9x^2 - 25$ b) $a^6 - 4b^6$

Solution: Because of the work already done in Example 7, it is a simple matter to factor (or "unmultiply") each of these problems.

a) $9x^2 - 25 = (3x + 5)(3x - 5)$

b) $a^6 - 4b^6 = (a^3 - 2b^3)(a^3 + 2b^3)$

In each case, note how we took the square roots of each term, then separated one set with a plus sign and the other with a minus sign. Because of the commutative property of multiplication, it does not matter which one you make plus and which one you make minus.

Answer: $(9x+7)(9x-7)$

□

Always remember the first rule of factoring.

The first rule of factoring. The first step to perform in any factoring problem is factor out the GCF.

You Try It!

Factor: $4x^4 - 16x^2$

EXAMPLE 9. Factor: $x^3 - 9x$

Solution: In $x^3 - 9x$, the GCF of x^3 and $9x$ is x. Factor out x.

$$x^3 - 9x = x(x^2 - 9)$$

Note that $x^2 - 9$ is now the difference of two perfect squares. Take the square roots of x^2 and 9, which are x and 3, then separate one set with a plus sign and the other set with a minus sign.

$$= x(x+3)(x-3)$$

Answer: $4x^2(x+2)(x-2)$

□

Factoring Completely

Sometimes after one pass at factoring, factors remain that can be factored further. You must continue to factor in this case.

You Try It!

Factor: $x^4 - 81$

EXAMPLE 10. Factor: $x^4 - 16$

Solution: In $x^4 - 16$, we have the difference of two squares: $(x^2)^2 = x^4$ and $4^2 = 16$. First, we take the square roots, then separate one set with a plus sign and the other set with a minus sign.

$$x^4 - 16 = (x^2+4)(x^2-4)$$

Note that x^2+4 is the *sum* of two squares and does not factor further. However, x^2-4 is the difference of two squares. Take the square roots, x and 2, then separate one set with a plus sign and the other set with a minus sign.

$$= (x^2+4)(x+2)(x-2)$$

Done. We cannot factor further.

Answer: $(x^2+9)(x+3)(x-3)$

□

Nonlinear Equations Revisited

Remember, if an equation is nonlinear, the first step is to make one side equal to zero by moving all terms to one side of the equation. Once you've completed this important first step, factor and apply the zero product property to find the solutions.

You Try It!

EXAMPLE 11. Solve for x: $25x^2 = 169$

Solve for x: $16x^2 = 121$

Solution: Make one side equal to zero, factor, then apply the zero product property.

$$\begin{aligned} 25x^2 &= 169 && \text{Original equation.} \\ 25x^2 - 169 &= 0 && \text{Subtract 169 from both sides.} \end{aligned}$$

Note that we have two perfect squares separated by a minus sign. This is the difference of squares pattern. Take the square roots, making one term plus and one term minus.

$$(5x+13)(5x-13) = 0 \qquad \text{Use difference of squares to factor.}$$

Use the zero product property to complete the solution, setting each factor equal to zero and solving the resulting equations.

$$\begin{aligned} 5x+13 &= 0 & \quad\text{or}\quad && 5x-13 &= 0 \\ x &= -\frac{13}{5} & && x &= \frac{13}{5} \end{aligned}$$

Hence, the solutions of $25x^2 = 169$ are $x = -13/5$ and $x = 13/5$. We encourage readers to check each of these solutions.

Answer: $-11/4$, $11/4$

□

You Try It!

Solve for x:
$25x^2 = 80x - 64$

EXAMPLE 12. Solve for x: $\quad 49x^2 + 81 = 126x$

Solution: Make one side equal to zero, factor, then apply the zero product property.

$$\begin{aligned} 49x^2 + 81 &= 126x && \text{Original equation.} \\ 49x^2 - 126x + 81 &= 0 && \text{Subtract } 126x \text{ from both sides.} \end{aligned}$$

Note that the first and last terms of the trinomial are perfect squares. Hence, it make sense to try and factor as a perfect square trinomial, taking the square roots of the first and last terms.

$$(7x - 9)^2 = 0 \qquad \text{Factor as a perfect square trinomial.}$$

Of course, be sure to check the middle term. Because $-2(7x)(9) = -126x$, the middle term is correct. Because $(7x - 9)^2 = (7x - 9)(7x - 9)$, we can use the zero product property to set each factor equal to zero and solve the resulting equations.

$$\begin{aligned} 7x - 9 &= 0 & \quad\text{or}\quad && 7x - 9 &= 0 \\ x &= \frac{9}{7} &&& x &= \frac{9}{7} \end{aligned}$$

Hence, the only solution of $49x^2 + 81 = 126x$ is $x = 9/7$. We encourage readers to check this solution.

One can also argue that the only number whose square is zero is the number zero. Hence, one can go directly from

$$(7x - 9)^2 = 0$$

to

$$7x - 9 = 0.$$

Hence, the only solution of $49x^2 + 81 = 126x$ is $x = 9/7$.

Answer: $8/5$

□

You Try It!

Solve for x:
$5x^3 + 36 = x^2 + 180x$

EXAMPLE 13. Solve for x: $\quad 2x^3 + 3x^2 = 50x + 75$

Solution: Make one side equal to zero, factor, then apply the zero product property.

$$\begin{aligned} 2x^3 + 3x^2 &= 50x + 75 && \text{Original equation.} \\ 2x^3 + 3x^2 - 50x - 75 &= 0 && \text{Make one side zero.} \end{aligned}$$

This is a four-term expression, so we try factoring by grouping. Factor x^2 out of the first two terms, and -25 out of the second two terms.

$$\begin{aligned} x^2(2x + 3) - 25(2x + 3) &= 0 && \text{Factor by grouping.} \\ (x^2 - 25)(2x + 3) &= 0 && \text{Factor out } 2x + 3. \end{aligned}$$

Complete the factorization by using the difference of squares to factor $x^2 - 25$.

$$(x+5)(x-5)(2x+3) = 0 \qquad \text{Use difference of squares to factor.}$$

Finally, use the zero product property. Set each factor equal to zero and solve for x.

$$x+5=0 \quad \text{or} \quad x-5=0 \quad \text{or} \quad 2x+3=0$$
$$x=-5 \qquad\qquad x=5 \qquad\qquad x=-\frac{3}{2}$$

Hence, the solutions of $2x^3+3x^2 = 50x+75$ are $x=-5$, $x=5$, and $x=-3/2$. We encourage readers to check each of these solutions.

Answer: -6, 6, $1/5$

□

Let's solve another nonlinear equation, matching the algebraic and graphical solutions.

You Try It!

Solve the equation $x^3 = 16x$ both algebraically and graphically, then compare your answers.

EXAMPLE 14. Solve the equation $x^3 = 4x$, both algebraically and graphically, then compare your answers.

Solution: Note that we have a power of x larger than one, so the equation $x^3 = 4x$ is nonlinear. Make one side zero and factor.

$$\begin{aligned} x^3 &= 4x && \text{Original equation.}\\ x^3 - 4x &= 0 && \text{Nonlinear. Make one side zero.}\\ x(x^2-4) &= 0 && \text{Factor out GCF.}\\ x(x+2)(x-2) &= 0 && \text{Apply difference of squares.} \end{aligned}$$

Note that we now have a product of three factors that equals zero. The zero product property says that at least one of these factors must equal zero.

$$x=0 \quad \text{or} \quad x+2=0 \quad \text{or} \quad x-2=0$$
$$x=-2 \qquad\qquad x=2$$

Hence, the solutions of $x^3 = 4x$ are $x=0$, $x=-2$, and $x=2$.

Graphical solution. Load $y = x^3$ and $y = 4x$ into **Y1** and **Y2** in the **Y=** menu of your calculator. Select **6:ZStandard** from the ZOOM menu to produce the graph in Figure 1.26.

Although the image in Figure 1.26 shows all three points of intersection, adjusting the WINDOW parameters as shown in Figure 1.27, then pressing the GRAPH button will produce a nicer view of the points of intersection, as shown in the figure on the right in Figures 1.27.

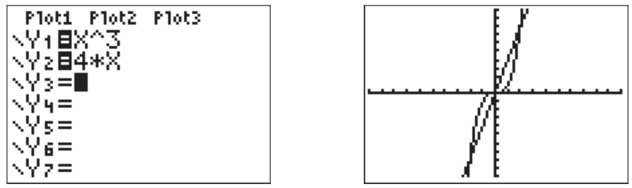


Figure 1.26: Sketching $y = x^3$ and $y = 4\,x$.

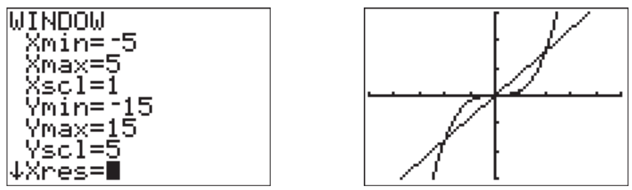


Figure 1.27: Adjusting the viewing window.

Use the **5:intersect** tool from the CALC menu to find the three points of intersection. Press the ENTER key in response to "First curve," then press ENTER again in response to "Second curve," then use the left-arrow key to move your cursor close to the leftmost point of intersection and press ENTER in response to "Guess." The result is shown in the first image on the left in Figure 1.28. Repeat the process to find the remaining points of intersection. The results are shown in the last two images in Figure 1.28.

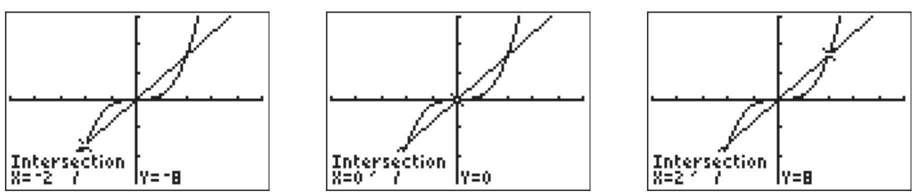


Figure 1.28: Finding the points of intersection.

Thus, the graphical solutions are $x = -2$, $x = 0$, and $x = 2$.

Reporting the solution on your homework: Duplicate the image in your calculator's viewing window on your homework page. Use a ruler to draw all lines, but freehand any curves.

- Label the horizontal and vertical axes with x and y, respectively (see Figure 1.29).
- Place your WINDOW parameters at the end of each axis (see Figure 1.29).
- Label the graph with its equation (see Figure 1.29).

- Drop dashed vertical lines through each x-intercept. Shade and label the x-values of the points where the dashed vertical line crosses the x-axis. These are the solutions of the equation $x^3 = 4x$ (see Figure 1.29).

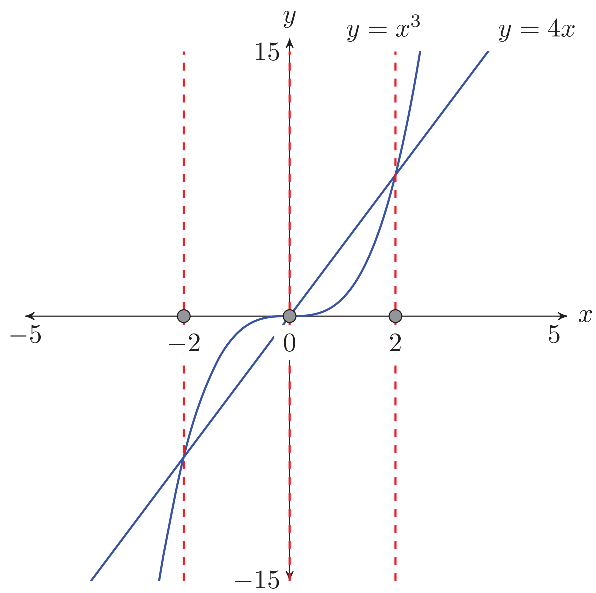
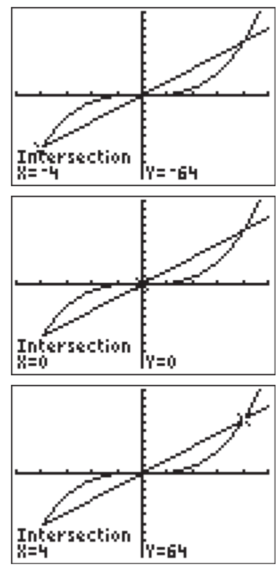


Figure 1.29: Reporting your graphical solution on your homework.

Finally, note that the graphical solutions $x = -2$, $x = 0$, and $x = 2$ match our algebraic solutions exactly.

Answer: -4, 0, 4



1.5 Exercises

In Exercises 1-8, expand each of the given expressions.

1. $(8r-3t)^2$

2. $(6a+c)^2$

3. $(4a+7b)^2$

4. $(4s+t)^2$

5. $(s^3-9)^2$

6. $(w^3+7)^2$

7. $(s^2+6t^2)^2$

8. $(7u^2-2w^2)^2$

In Exercises 9-28, factor each of the given expressions.

9. $25s^2+60st+36t^2$

10. $9u^2+24uv+16v^2$

11. $36v^2-60vw+25w^2$

12. $49b^2-42bc+9c^2$

13. $a^4+18a^2b^2+81b^4$

14. $64u^4-144u^2w^2+81w^4$

15. $49s^4-28s^2t^2+4t^4$

16. $4a^4-12a^2c^2+9c^4$

17. $49b^6-112b^3+64$

18. $25x^6-10x^3+1$

19. $49r^6+112r^3+64$

20. a^6-16a^3+64

21. $5s^3t-20s^2t^2+20st^3$

22. $12r^3t-12r^2t^2+3rt^3$

23. $8a^3c+8a^2c^2+2ac^3$

24. $18x^3z-60x^2z^2+50xz^3$

25. $-48b^3+120b^2-75b$

26. $-45c^3+120c^2-80c$

27. $-5u^5-30u^4-45u^3$

28. $-12z^5-36z^4-27z^3$

In Exercises 29-36, expand each of the given expressions.

29. $(21c+16)(21c-16)$

30. $(19t+7)(19t-7)$

31. $(5x+19z)(5x-19z)$

32. $(11u+5w)(11u-5w)$

33. $(3y^4+23z^4)(3y^4-23z^4)$

34. $(5x^3+z^3)(5x^3-z^3)$

35. $(8r^5+19s^5)(8r^5-19s^5)$

36. $(3u^3+16v^3)(3u^3-16v^3)$

In Exercises 37-60, factor each of the given expressions.

37. $361x^2 - 529$

38. $9b^2 - 25$

39. $16v^2 - 169$

40. $81r^2 - 169$

41. $169x^2 - 576y^2$

42. $100y^2 - 81z^2$

43. $529r^2 - 289s^2$

44. $49a^2 - 144b^2$

45. $49r^6 - 256t^6$

46. $361x^{10} - 484z^{10}$

47. $36u^{10} - 25w^{10}$

48. $a^6 - 81c^6$

49. $72y^5 - 242y^3$

50. $75y^5 - 147y^3$

51. $1444a^3b - 324ab^3$

52. $12b^3c - 1875bc^3$

53. $576x^3z - 1156xz^3$

54. $192u^3v - 507uv^3$

55. $576t^4 - 4t^2$

56. $4z^5 - 256z^3$

57. $81x^4 - 256$

58. $81x^4 - 1$

59. $81x^4 - 16$

60. $x^4 - 1$

In Exercises 61-68, factor each of the given expressions completely.

61. $z^3 + z^2 - 9z - 9$

62. $3u^3 + u^2 - 48u - 16$

63. $x^3 - 2x^2y - xy^2 + 2y^3$

64. $x^3 + 2x^2z - 4xz^2 - 8z^3$

65. $r^3 - 3r^2t - 25rt^2 + 75t^3$

66. $2b^3 - 3b^2c - 50bc^2 + 75c^3$

67. $2x^3 + x^2 - 32x - 16$

68. $r^3 - 2r^2 - r + 2$

In Exercises 69-80, solve each of the given equations for x.

69. $2x^3 + 7x^2 = 72x + 252$

70. $2x^3 + 7x^2 = 32x + 112$

71. $x^3 + 5x^2 = 64x + 320$

72. $x^3 + 4x^2 = 49x + 196$

73. $144x^2 + 121 = 264x$

74. $361x^2 + 529 = 874x$

75. $16x^2 = 169$

76. $289x^2 = 4$

77. $9x^2 = 25$

78. $144x^2 = 121$

79. $256x^2 + 361 = -608x$

80. $16x^2 + 289 = -136x$

In Exercises 81-84, perform each of the following tasks:

i) Use a strictly algebraic technique to solve the given equation.
ii) Use the **5:intersect** utility on your graphing calculator to solve the given equation. Report the results found using graphing calculator as shown in Example 12.

81. $x^3 = x$

82. $x^3 = 9x$

83. $4x^3 = x$

84. $9x^3 = x$

1.5 *Answers*

1. $64r^2 - 48rt + 9t^2$

3. $16a^2 + 56ab + 49b^2$

5. $s^6 - 18s^3 + 81$

7. $s^4 + 12s^2t^2 + 36t^4$

9. $(5s + 6t)^2$

11. $(6v - 5w)^2$

13. $(a^2 + 9b^2)^2$

15. $(7s^2 - 2t^2)^2$

17. $(7b^3 - 8)^2$

19. $(7r^3 + 8)^2$

21. $5st(s - 2t)^2$

23. $2ac(2a + c)^2$

25. $-3b(4b - 5)^2$

27. $-5u^3(u + 3)^2$

29. $441c^2 - 256$

31. $25x^2 - 361z^2$

33. $9y^8 - 529z^8$

35. $64r^{10} - 361s^{10}$

37. $(19x + 23)(19x - 23)$

39. $(4v + 13)(4v - 13)$

41. $(13x + 24y)(13x - 24y)$

43. $(23r + 17s)(23r - 17s)$

45. $(7r^3 + 16t^3)(7r^3 - 16t^3)$

47. $(6u^5 + 5w^5)(6u^5 - 5w^5)$

49. $2y^3(6y + 11)(6y - 11)$

51. $4ab(19a + 9b)(19a - 9b)$

53. $4xz(12x + 17z)(12x - 17z)$

55. $4t^2(12t + 1)(12t - 1)$

57. $(9x^2 + 16)(3x + 4)(3x - 4)$

59. $(9x^2 + 4)(3x + 2)(3x - 2)$

61. $(z + 3)(z - 3)(z + 1)$

63. $(x + y)(x - y)(x - 2y)$

65. $(r + 5t)(r - 5t)(r - 3t)$

67. $(x + 4)(x - 4)(2x + 1)$

69. $x = -6,\ 6,\ -\frac{7}{2}$

71. $x = -8,\ 8,\ -5$

73. $x = \frac{11}{12}$

75. $x = -\frac{13}{4}, \frac{13}{4}$

77. $x = -\frac{5}{3}, \frac{5}{3}$

79. $x = -\frac{19}{16}$

81. $x = 0, -1, 1$

83. $x = 0, -1/2, 1/2$

1.6 Applications of Factoring

In this section we will solve applications whose solutions involve factoring. Let's begin.

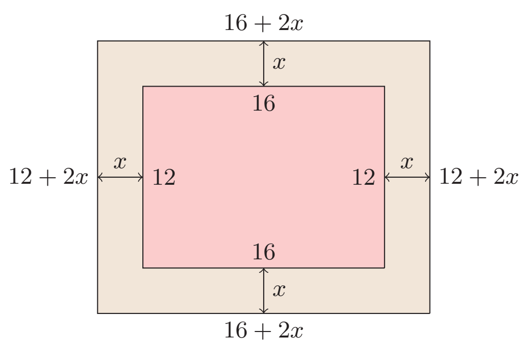
You Try It!

A rectangular canvas picture measures 7 inches by 11 inches. The canvas is mounted inside a frame of uniform width, increasing the total area covered by both canvas and frame to 117 square inches. Find the uniform width of the frame.

EXAMPLE 1. A rectangular canvas picture measures 12 inches by 16 inches. The canvas is mounted inside a frame of uniform width, increasing the total area covered by both canvas and frame to 396 square inches. Find the uniform width of the frame.

Solution: We follow the *Requirements for Word Problem Solutions*.

1. *Set up a variable dictionary.* A carefully labeled figure will help us maintain our focus. We'll let x represent the uniform width of the frame.



2. *Set up an equation.* If the inner rectangular dimensions are 16 inches by 12 inches, adding a frame of uniform width x makes the dimensions of the frame plus canvas $16 + 2x$ inches by $12 + 2x$ inches. The total area is found by multiplying these outer dimensions, $A = (16 + 2x)(12 + 2x)$. If the total area is $A = 396$ square inches, then we have the following equation.
$$(16 + 2x)(12 + 2x) = 396$$

3. *Solve the equation.* We start by expanding the right-hand side of the equation.
$$(16 + 2x)(12 + 2x) = 396$$
$$192 + 56x + 4x^2 = 396$$
The resulting equation is nonlinear. Make one side zero.
$$4x^2 + 56x - 204 = 0$$

We could factor out 4 on the left-hand side, but since there is a zero on the right-hand side of the equation, it's a bit easier to simply divide both sides by 4. Note how we divide each term on the left-hand side by the number 4.

$$x^2 + 14x - 51 = 0$$

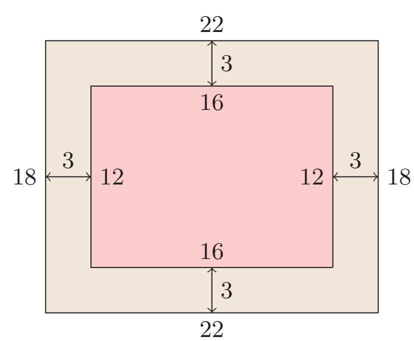
We need an integer pair whose product is $ac = -51$ and whose sum is $b = 14$. The integer pair $-3, 17$ comes to mind. Since the coefficient of x^2 is one, we can simply "drop in place" our ordered pair.

$$(x - 3)(x + 17) = 0$$

Thus, the solutions are $x = 3$ and $x = -17$.

4. *Answer the question.* The uniform width of the frame cannot be -17 inches, so we eliminate this solution from consideration. Hence, the uniform width of the frame is 3 inches.

5. *Look back.* If the uniform width of the frame is 3 inches, then the final dimensions will look like the following.



Thus, the combined area of the frame plus canvas is $A = (18)(22)$, or $A = 396$ square inches, the area given in the problem statement. Our solution is correct.

Answer: 1 inch

□

You Try It!

A projectile's height (in feet) is given by the equation $y = -16t^2 + 144t + 576$, where time t is measured in seconds. How much time passes before the projectile hits the ground?

EXAMPLE 2. A projectile is fired at an angle into the air from atop a cliff overlooking the ocean. The projectile's distance (in feet) from the base of the cliff is give by the equation

$$x = 120t, \tag{6.1}$$

and the projectile's height above sea level (in feet) is given by the equation

$$y = -16t^2 + 288t + 640, \tag{6.2}$$

where t is the amount of time (in seconds) that has passed since the projectile's release.

a) How much time passes before the projectile splashes into the ocean?

b) At that time, how far is the projectile from the base of the cliff?

Solution: We follow the *Requirements for Word Problem Solutions*.

1. *Set up a variable dictionary.* The variables are already set up for us.

$$\begin{aligned} x &= \text{Distance from base of the cliff (in feet).} \\ y &= \text{Height above sea level (in feet).} \\ t &= \text{Time since projectile's release (in seconds).} \end{aligned}$$

2. *Set up an equation.* The equations are already set up (see equation 1.1 and equation 1.2).

3. *Solve the equation.* When the projectile splashes into the ocean, its height above sea level at that moment is $y = 0$ feet. Substitute 0 for y in equation 1.2 and solve the resulting equation for t.

$$\begin{aligned} y &= -16t^2 + 288t + 640 \\ 0 &= -16t^2 + 288t + 640 \end{aligned}$$

We could factor out -16, but since the left-hand side of this equation is zero, it's a bit easier to divide both sides by -16. Note how we divide each term on the right-hand side by -16.

$$0 = t^2 - 18t - 40$$

We need a pair of integers so that their product is $ac = -40$ and their sum is -18. The integer pair $2, -20$ comes to mind. Since the coefficient of t^2 is one, we can "drop in place" our integer pair.

$$0 = (t + 2)(t - 20)$$

Hence, the solutions are $t = -2$ and $t = 20$.

4. *Answer the question.* In answering question (a), the solution $t = -2$ seconds makes no sense. Thus, the projectile splashes into the ocean at $t = 20$ seconds.

In addressing question (b), to find the projectile's distance from the base of the cliff at this moment, substitute $t = 20$ in equation 1.1.

$$\begin{aligned} x &= 120t \\ x &= 120(20) \\ x &= 2400 \end{aligned}$$

Hence, at the moment the projectile splashes into the ocean, it is 2,400 feet from the base of the cliff.

5. *Look back.* The best we can do here is check our solution $t = 20$ in equation 1.2.

$$\begin{aligned} y &= -16t^2 + 288t + 640 \\ y &= -16(20)^2 + 288(20) + 640 \\ y &= -6400 + 5760 + 640 \\ y &= 0 \end{aligned}$$

Indeed, at $t = 20$, the projectile does splash into the ocean.

Answer: 12 seconds

□

You Try It!

The product of two consecutive positive odd integers is 483. Find the integers.

EXAMPLE 3. The product of two consecutive even integers is 728. Find the integers.

Solution: We follow the *Requirements for Word Problem Solutions.*

1. *Set up a variable dictionary.* Let k represent the first even integer. Then $k + 2$ represents the next consecutive even integer.

2. *Set up an equation.* The product of the integers is 728. Hence, we have the following equation.

$$k(k + 2) = 728$$

3. *Solve the equation.* Expand the left-hand side of the equation.

$$k^2 + 2k = 728$$

The equation is nonlinear. Make one side zero.

$$k^2 + 2k - 728 = 0$$

See Using the Calculator to Assist the *ac*-Method on page 462. We need an integer pair whose product is $ac = -728$ and whose sum is $b = 2$. Enter **-728/X** in **Y1**, then set up the table (see Figure 1.31).

Figure 1.31: Load ac/X, o r **-728/X** into Y1 in the Y= menu.

Use the up- and down-arrow keys to scroll. Note that $28, -26$ is the desired pair. Because the coefficient of k^2 is one, we can "drop in place" the ordered pair.

$$0 = (k + 28)(k - 26)$$

Hence, the solutions are $k = -28$ and $k = 26$.

4. *Answer the question.* If $k = -28$, the next consecutive even integer is $k + 2 = -26$. Secondly, if $k = 26$, the next consecutive even integer is $k + 2 = 28$.

5. *Look back.* Our first pair is -28 and -26. They have the required product of 728. Our second pair is 26 and 28. Their product is also 728. Both solutions check!

Answer: 21 and 23

□

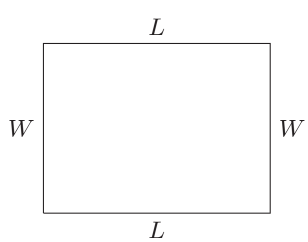
You Try It!

A rectangle has perimeter 62 feet and area 234 square feet. Find the dimensions of the rectangle.

EXAMPLE 4. A rectangle has perimeter 54 feet and area 180 square feet. Find the dimensions of the rectangle.

Solution: We follow the *Requirements for Word Problem Solutions.*

1. *Set up a variable dictionary.* A sketch will help us keep our focus. Let L represent the length of the rectangle and let W represent its width.



2. *Set up an equation.* The perimeter is 54 feet, so $2W+2L=54$, or dividing both sides by 2:
$$W+L=27 \tag{1.3}$$
The area is 180 square feet, so:
$$LW=180 \tag{1.4}$$

3. *Solve the equation.* The system of equations (equations 1.3 and 1.4)can be solved using the substitution method. First, solve equation 1.3 for W:
$$W=27-L \tag{1.5}$$
Substitute equation 1.5 in equation 1.4, expand, then make one side zero.
$$\begin{aligned} L(27-L) &= 180 \\ 27L-L^2 &= 180 \\ 0 &= L^2-27L+180 \end{aligned}$$
The integer pair $-12, -15$ has product $ac=180$ and sum $b=-27$. Moreover, the coefficient of L^2 is one, so we can "drop in place" our integer pair.
$$0=(L-12)(L-15)$$
Hence, the solutions are $L=12$ and $L=15$.

4. *Answer the question.* Two possibilities for the width.

Substitute $L=$12in 1.5).

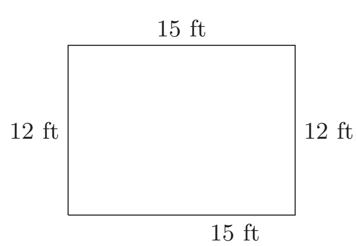
$$\begin{aligned} W &= 27-L \\ W &= 27-12 \\ W &= 15 \end{aligned}$$

Substitute $L=$15 in 1.5).

$$\begin{aligned} W &= 27-L \\ W &= 27-15 \\ W &= 12 \end{aligned}$$

Both answers give the same 15 by 12 foot rectangle, but we usually think of the term "length" as the longer side of the rectangle. So let's go with the length is $L=15$ feet and the width is $W=12$ feet.

5. *Look back.* Let's add $L=15$ feet and $W=12$ feet to a diagram.



If we add the dimensions around the rectangle, the perimeter is $P=15+12+15+12$, or $P=54$ feet, the perimeter required in the problem statement. Next, if we multiply the dimensions, then $A=(15)(12)$, or $A=180$ square feet, the area required in the problem statement. Our solution is correct!

Answer: length = 18 feet and width = 13 feet

□

1.6 *Exercises*

1. A rectangular canvas picture measures 14 inches by 36 inches. The canvas is mounted inside a frame of uniform width, increasing the total area covered by both canvas and frame to 720 square inches. Find the uniform width of the frame.

2. A rectangular canvas picture measures 10 inches by 32 inches. The canvas is mounted inside a frame of uniform width, increasing the total area covered by both canvas and frame to 504 square inches. Find the uniform width of the frame.

3. A projectile is fired at an angle into the air from atop a cliff overlooking the ocean. The projectile's distance (in feet) from the base of the cliff is give by the equation

$$x = 180t,$$

and the projectile's height above sea level (in feet) is given by the equation

$$y = -16t^2 + 352t + 1664,$$

where t is the amount of time (in seconds) that has passed since the projectile's release. How much time passes before the projectile splashes into the ocean? At that time, how far is the projectile from the base of the cliff?

4. A projectile is fired at an angle into the air from atop a cliff overlooking the ocean. The projectile's distance (in feet) from the base of the cliff is give by the equation

$$x = 140t,$$

and the projectile's height above sea level (in feet) is given by the equation

$$y = -16t^2 + 288t + 1408,$$

where t is the amount of time (in seconds) that has passed since the projectile's release. How much time passes before the projectile splashes into the ocean? At that time, how far is the projectile from the base of the cliff?

5. The product of two consecutive even integers is 624. Find the integers.

6. The product of two consecutive even integers is 528. Find the integers.

7. The product of two consecutive positive integers is 552. Find the integers.

8. The product of two consecutive positive integers is 756. Find the integers.

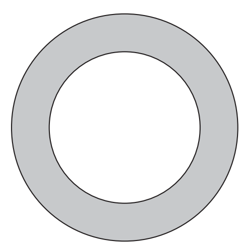
9. The product of two consecutive odd integers is 483. Find the integers.

10. The product of two consecutive odd integers is 783. Find the integers.

11. A rectangle has perimeter 42 feet and area 104 square feet. Find the dimensions of the rectangle.

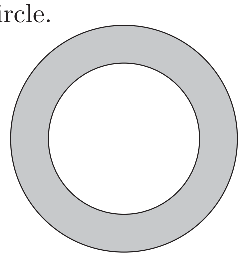
12. A rectangle has perimeter 32 feet and area 55 square feet. Find the dimensions of the rectangle.

13. The radius of the outer circle is one inch longer than twice the radius of the inner circle.



If the area of the shaded region is 40π square inches, what is the length of the inner radius?

14. The radius of the outer circle is two inches longer than three times the radius of the inner circle.



If the area of the shaded region is 180π square inches, what is the length of the inner radius?

15. You have two positive numbers. The second number is three more than two times the first number. The difference of their squares is 144. Find both positive numbers.

16. You have two positive numbers. The second number is two more than three times the first number. The difference of their squares is 60. Find both positive numbers.

17. Two numbers differ by 5. The sum of their squares is 97. Find the two numbers.

18. Two numbers differ by 6. The sum of their squares is 146. Find the two numbers.

19. The length of a rectangle is three feet longer than six times its width. If the area of the rectangle is 165 square feet, what is the width of the rectangle?

20. The length of a rectangle is three feet longer than nine times its width. If the area of the rectangle is 90 square feet, what is the width of the rectangle?

21. The ratio of the width to the length of a given rectangle is 2 to 3, or $2/3$. If the width and length are both increased by 4 inches, the area of the resulting rectangle is 80 square inches. Find the width and length of the original rectangle.

22. The ratio of the width to the length of a given rectangle is 3 to 4, or $3/4$. If the width is increased by 3 inches and the length is increased by 6 inches, the area of the resulting rectangle is 126 square inches. Find the width and length of the original rectangle.

1.6 Answers

1. 2 inches

3. 26 seconds, 4,680 feet

5. -26 and -24, and 24 and 26

7. 23, 24

9. -23 and -21, and 21 and 23

11. 8 feet by 13 feet

13. 3 inches

15. 5 and 13

17. 4 and 9, and -4 and -9

19. 5 feet

21. 4 inches by 6 inches

Chapter 2 Quadratic Functions

In this chapter we study one of the most famous of mathematical concepts–the *parabola*. The most basic parabola is shaped rather like a "U," as shown in the margin. Whereas the graphs of linear functions like $f(x) = mx + b$ are lines, the graphs of functions having the form

A parabola.

$$f(x) = ax^2 + bx + c, \tag{1}$$

where a, b, and c are arbitrary numbers, are parabolas. These functions are called *quadratic functions*.

Apollonius (262 BC to 190 BC) wrote the quintessential text on the conic sections–of which the parabola is one–and is credited with giving the parabola its name.

In nature, approximations of parabolas are found in many diverse situations. Early in the 17th century, the parabolic trajectory of projectiles was discovered experimentally by Galileo (1564 to 1642), who performed experiments with balls rolling on inclined planes. The parabolic shape for projectiles was later proven mathematically by Isaac Newton (1643 to 1727). He found that, if we assume that there is no air resistance, parabolas can be used to model the trajectory of a body in motion under the influence of gravity (for instance, a rock flying through the air, neglecting air friction). We will study this application in detail in Section 5.5.

Parabolic arches in Las Vegas fountains.

Other applications of parabolas include the modeling of suspension bridges; the shapes of satellite dishes, heaters, and automobile headlights; braking distance and stopping distance of cars; and the path of water projected from a fountain, like at the water show at the Bellagio Hotel in Las Vegas.

2.1 The Parabola

In this section you will learn how to draw the graph of the quadratic function defined by the equation

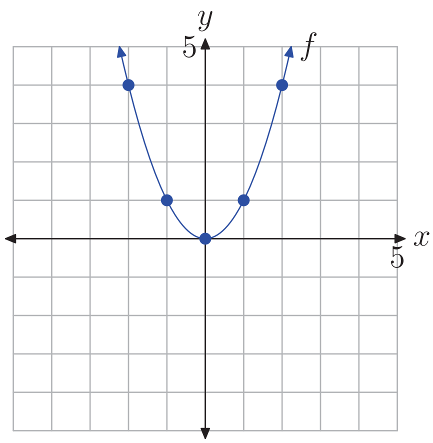
$$f(x) = a(x-h)^2 + k. \tag{1}$$

You will quickly learn that the graph of the quadratic function is shaped like a "U" and is called a *parabola*. The form of the quadratic function in **equation (1)** is called *vertex form*, so named because the form easily reveals the *vertex* or "turning point" of the parabola. Each of the constants in the vertex form of the quadratic function plays a role. As you will soon see, the constant a controls the scaling (stretching or compressing of the parabola), the constant h controls a horizontal shift and placement of the *axis of symmetry*, and the constant k controls the vertical shift.

Let's begin by looking at the *scaling* of the quadratic.

Scaling the Quadratic

The graph of the basic quadratic function $f(x) = x^2$ shown in **Figure 1**(a) is called a *parabola*. We say that the parabola in **Figure 1**(a) "opens upward." The point at $(0,0)$, the "turning point" of the parabola, is called the *vertex* of the parabola. We've tabulated a few points for reference in the table in **Figure 1**(b) and then superimposed these points on the graph of $f(x) = x^2$ in **Figure 1**(a).



(a) A basic parabola.

x	$f(x) = x^2$
-2	4
-1	1
0	0
1	1
2	4

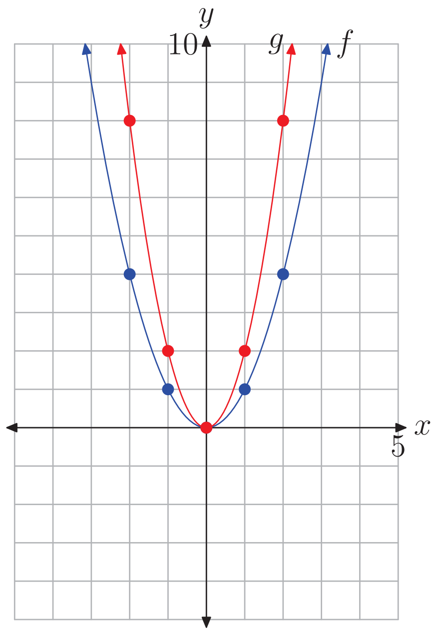
(b) Table of x-values and function values satisfying $f(x) = x^2$.

Figure 1. The graph of the basic parabola is a fundamental starting point.

Now that we know the basic shape of the parabola determined by $f(x) = x^2$, let's see what happens when we scale the graph of $f(x) = x^2$ in the vertical direction. For

[1] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

example, let's investigate the graph of $g(x) = 2x^2$. The factor of 2 has a doubling effect. Note that each of the function values of g is twice the corresponding function value of f in the table in **Figure 2**(b).



(a) The graphs of f and g.

x	$f(x) = x^2$	$g(x) = 2x^2$
-2	4	8
-1	1	2
0	0	0
1	1	2
2	4	8

(b) Table of x-values and function values satisfying $f(x) = x^2$ and $g(x) = 2x^2$.

Figure 2. A stretch by a factor of 2 in the vertical direction.

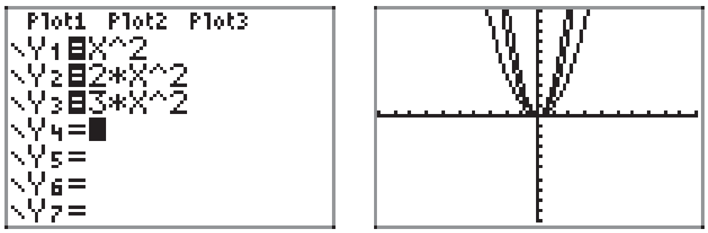
When the points in the table in **Figure 2**(b) are added to the coordinate system in **Figure 2**(a), the resulting graph of g is stretched by a factor of two in the vertical direction. It's as if we had put the original graph of f on a sheet of rubber graph paper, grabbed the top and bottom edges of the sheet, and then pulled each edge in the vertical direction to stretch the graph of f by a factor of two. Consequently, the graph of $g(x) = 2x^2$ appears somewhat narrower in appearance, as seen in comparison to the graph of $f(x) = x^2$ in **Figure 2**(a). Note, however, that the vertex at the origin is unaffected by this scaling.

In like manner, to draw the graph of $h(x) = 3x^2$, take the graph of $f(x) = x^2$ and *stretch* the graph by a factor of three, tripling the y-value of each point on the original graph of f. This idea leads to the following result.

Property 2. If a is a constant larger than 1, that is, if $a > 1$, then the graph of $g(x) = ax^2$, when compared with the graph of $f(x) = x^2$, is *stretched* by a factor of a.

I **Example 3.** *Compare the graphs of $y = x^2$, $y = 2x^2$, and $y = 3x^2$ on your graphing calculator.*

Load the functions $y = x^2$, $y = 2x^2$, and $y = 3x^2$ into the `Y=` menu, as shown in **Figure 3**(a). Push the `ZOOM` button and select `6:ZStandard` to produce the image shown in **Figure 3**(b).

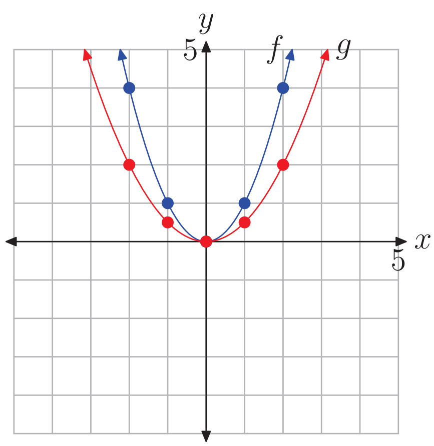


(a) (b)

Figure 3. Drawing $y = x^2$, $y = 2x^2$, and $y = 3x^2$ on the graphing calculator.

Note that as the "a" in $y = ax^2$ increases from 1 to 2 to 3, the graph of $y = ax^2$ stretches further and becomes, in a sense, narrower in appearance.

Next, let's consider what happens when we scale by a number that is smaller than 1 (but greater than zero — we'll deal with the negative in a moment). For example, let's investigate the graph of $g(x) = (1/2)x^2$. The factor 1/2 has a halving effect. Note that each of the function values of g is half the corresponding function value of f in the table in **Figure 4**(b).



(a) The graphs of f and g.

x	$f(x) = x^2$	$g(x) = (1/2)x^2$
−2	4	2
−1	1	1/2
0	0	0
1	1	1/2
2	4	2

(a) Table of x-values and function values satisfying $f(x) = x^2$ and $g(x) = (1/2)x^2$.

Figure 4. A compression by a factor of 2 in the vertical direction.

When the points in the table in **Figure 4**(b) are added to the coordinate system in **Figure 4**(a), the resulting graph of g is compressed by a factor of 2 in the vertical direction. It's as if we again placed the graph of $f(x) = x^2$ on a sheet of rubber graph paper, grabbed the top and bottom of the sheet, and then *squeezed* them together by a factor of two. Consequently, the graph of $g(x) = (1/2)x^2$ appears somewhat wider in appearance, as seen in comparison to the graph of $f(x) = x^2$ in **Figure 4**(a). Note again that the vertex at the origin is unaffected by this scaling.

Property 4. If a is a constant smaller than 1 (but larger than zero), that is, if $0 < a < 1$, then the graph of $g(x) = ax^2$, when compared with the graph of $f(x) = x^2$, is *compressed* by a factor of $1/a$.

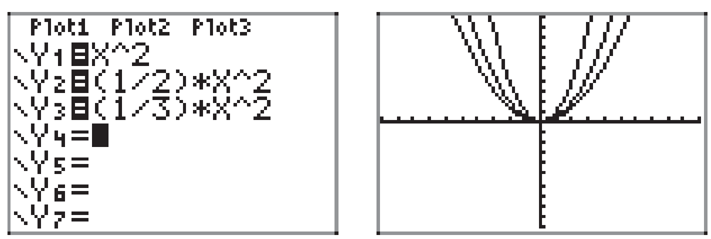
Some find **Property 4** somewhat counterintuitive. However, if you compare the function $g(x) = (1/2)x^2$ with the general form $g(x) = ax^2$, you see that $a = 1/2$. Property **4** states that the graph will be compressed by a factor of $1/a$. In this case, $a = 1/2$ and

$$\frac{1}{a} = \frac{1}{1/2} = 2.$$

Thus, **Property 4** states that the graph of $g(x) = (1/2)x^2$ should be compressed by a factor of $1/(1/2)$ or 2, which is seen to be the case in **Figure 4**(a).

I **Example 5.** *Compare the graphs of $y = x^2$, $y = (1/2)x^2$, and $y = (1/3)x^2$ on your graphing calculator.*

Load the equations $y = x^2$, $y = (1/2)x^2$, and $y = (1/3)x^2$ into the `Y=`, as shown in **Figure 2**(a). Push the `ZOOM` button and select `6:ZStandard` to produce the image shown in **Figure 2**(b).



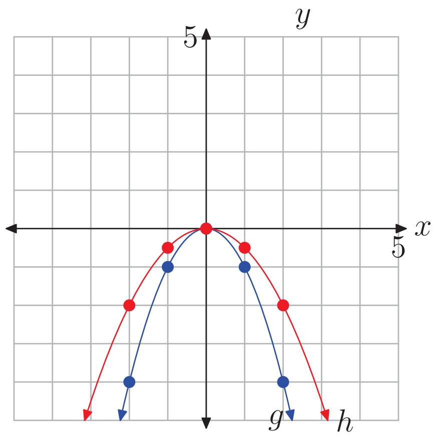
(a) (b)

Figure 2. Drawing $y = x^2$, $y = (1/2)x^2$, and $y = (1/3)x^2$ on the graphing calculator.

Note that as the "a" in $y = ax^2$ decreases from 1 to 1/2 to 1/3, the graph of $y = ax^2$ compresses further and becomes, in a sense, wider in appearance.

Vertical Reflections

Let's consider the graph of $g(x) = ax^2$, when $a < 0$. For example, consider the graphs of $g(x) = -x^2$ and $h(x) = (-1/2)x^2$ in **Figure 6**.



(a) The graphs of g and h.

x	$g(x) = -x^2$	$h(x) = (-1/2)x^2$
-2	-4	-2
-1	-1	$-1/2$
0	0	0
1	-1	$-1/2$
2	-4	-2

(b) Table of x-values and function values satisfying $g(x) = -x^2$ and $h(x) = (-1/2)x^2$.

Figure 6. A vertical reflection across the x-axis.

When the table in **Figure 6**(b) is compared with the table in **Figure 4**(b), it is easy to see that the numbers in the last two columns are the same, but they've been negated. The result is easy to see in **Figure 6**(a). The graphs have been reflected across the x-axis. Each of the parabolas now "opens downward."

However, it is encouraging to see that the scaling role of the constant a in $g(x) = ax^2$ has not changed. In the case of $h(x) = (-1/2)x^2$, the y-values are still "compressed" by a factor of two, but the minus sign negates these values, causing the graph to reflect across the x-axis. Thus, for example, one would think that the graph of $y = -2x^2$ would be *stretched* by a factor of two, then reflected across the x-axis. Indeed, this is correct, and this discussion leads to the following property.

Property 6. If $-1 < a < 0$, then the graph of $g(x) = ax^2$, when compared with the graph of $f(x) = x^2$, is compressed by a factor of $1/|a|$, then reflected across the x-axis. Secondly, if $a < -1$, then the graph of $g(x) = ax^2$, when compared with the graph of $f(x) = x^2$, is stretched by a factor of $|a|$, then reflected across the x-axis.

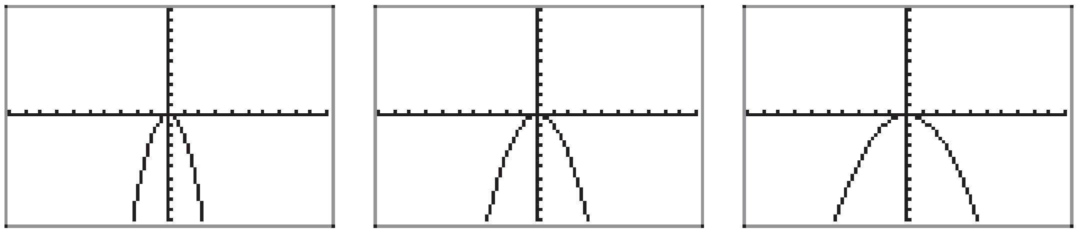
Again, some find **Property 6** confusing. However, if you compare $g(x) = (-1/2)x^2$ with the general form $g(x) = ax^2$, you see that $a = -1/2$. Note that in this case, $-1 < a < 0$. **Property 6** states that the graph will be compressed by a factor of $1/|a|$. In this case, $a = -1/2$ and

$$\frac{1}{|a|} = \frac{1}{|-1/2|} = 2.$$

That is, **Property 6** states that the graph of $g(x) = (-1/2)x^2$ is compressed by a factor of $1/(|-1/2|)$, or 2, then reflected across the x-axis, which is seen to be the case in **Figure 6**(a). Note again that the vertex at the origin is unaffected by this scaling and reflection.

I **Example 7.** *Sketch the graphs of $y = -2x^2$, $y = -x^2$, and $y = (-1/2)x^2$ on your graphing calculator.*

Each of the equations were loaded separately into `Y1` in the `Y=` menu. In each of the images in **Figure 7**, we selected `6:ZStandard` from the `ZOOM` menu to produce the image.



(a) $y = -2x^2$ (b) $y = -x^2$ (c) $y = (-1/2)x^2$

Figure 7.

In **Figure 7**(b), the graph of $y = -x^2$ is a reflection of the graph of $y = x^2$ across the x-axis and opens downward. In **Figure 7**(a), note that the graph of $y = -2x^2$ is stretched vertically by a factor of 2 (compare with the graph of $y = -x^2$ in **Figure 7**(b)) and reflected across the x-axis to open downward. In **Figure 7**(c), the graph of $(-1/2)x^2$ is compressed by a factor of 2, appears a bit wider, and is reflected across the x-axis to open downward.

Horizontal Translations

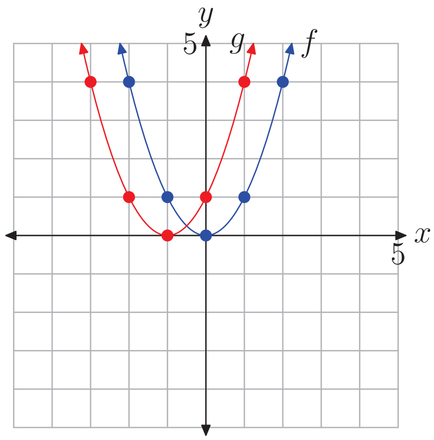
The graph of $g(x) = (x + 1)^2$ in **Figure 8**(a) shows a basic parabola that is shifted one unit to the left. Examine the table in **Figure 8**(b) and note that the equation $g(x) = (x + 1)^2$ produces the same y-values as does the equation $f(x) = x^2$, the only difference being that these y-values are calculated at x-values that are one unit less than those used for $f(x) = x^2$. Consequently, the graph of $g(x) = (x + 1)^2$ must shift one unit to the left of the graph of $f(x) = x^2$, as is evidenced in **Figure 8**(a).

Note that this result is counterintuitive. One would think that replacing x with $x + 1$ would shift the graph one unit to the right, but the shift actually occurs in the opposite direction.

Finally, note that this time the vertex of the parabola has shifted 1 unit to the left and is now located at the point $(-1, 0)$.

We are led to the following conclusion.

Property 8. If $c > 0$, then the graph of $g(x) = (x + c)^2$ is shifted c units to the left of the graph of $f(x) = x^2$.



(a) The graphs of f and g.

x	$f(x) = x^2$	x	$g(x) = (x+1)^2$
-2	4	-3	4
-1	1	-2	1
0	0	-1	0
1	1	0	1
2	4	1	4

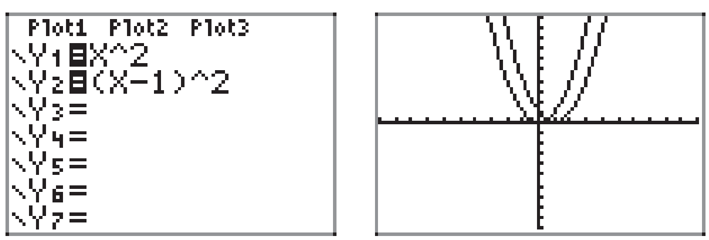
(a) Table of x-values and function values satisfying $f(x) = x^2$ and $g(x) = (x+1)^2$.

Figure 8. A horizontal shift or translation.

A similar thing happens when you replace x with $x - 1$, only this time the graph is shifted one unit to the right.

I **Example 9.** *Sketch the graphs of $y = x^2$ and $y = (x-1)^2$ on your graphing calculator.*

Load the equations $y = x^2$ and $y = (x-1)^2$ into the `Y=` menu, as shown in **Figure 9**(a). Push the `ZOOM` button and select `6:ZStandard` to produce the image shown in **Figure 9**(b).



(a) (b)

Figure 9. Drawing $y = x^2$ and $y = (x-1)^2$ on the graphing calculator.

Note that the graph of $y = (x-1)^2$ is shifted 1 unit to the right of the graph of $y = x^2$ and the vertex of the graph of $y = (x-1)^2$ is now located at the point $(1, 0)$.

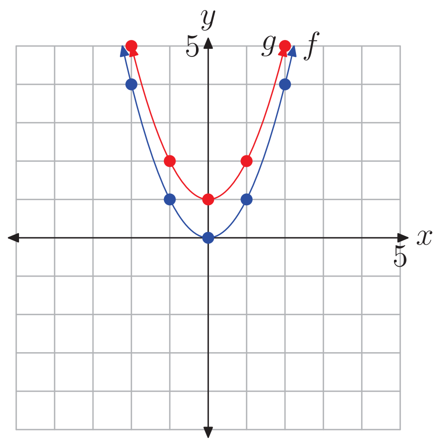
We are led to the following property.

Property 10. If $c > 0$, then the graph of $g(x) = (x-c)^2$ is shifted c units to the right of the graph of $f(x) = x^2$.

Vertical Translations

The graph of $g(x) = x^2 + 1$ in **Figure 10**(a) is shifted one unit upward from the graph of $f(x) = x^2$. This is easy to see as both equations use the same x-values in the table in **Figure 10**(b), but the function values of $g(x) = x^2 + 1$ are one unit larger than the corresponding function values of $f(x) = x^2$.

Note that the vertex of the graph of $g(x) = x^2 + 1$ has also shifted upward 1 unit and is now located at the point $(0, 1)$.



x	$f(x) = x^2$	$g(x) = x^2 + 1$
−2	4	5
−1	1	2
0	0	1
1	1	2
2	4	5

Figure 10. A vertical shift or translation.

The above discussion leads to the following property.

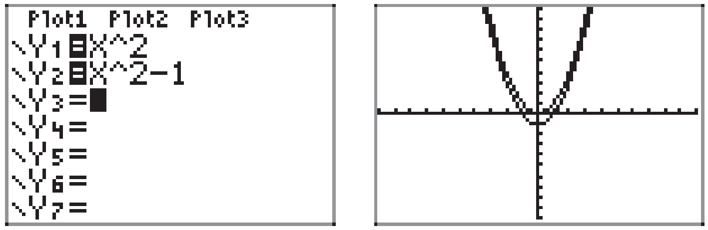
Property 11. If $c > 0$, the graph of $g(x) = x^2 + c$ is shifted c units upward from the graph of $f(x) = x^2$.

In a similar vein, the graph of $y = x^2 - 1$ is shifted downward one unit from the graph of $y = x^2$.

I **Example 12.** *Sketch the graphs of $y = x^2$ and $y = x^2 - 1$ on your graphing calculator.*

Load the equations $y = x^2$ and $y = x^2 - 1$ into the `Y=` menu, as shown in **Figure 11**(a). Push the `ZOOM` button and select `6:ZStandard` to produce the image shown in **Figure 11**(b).

Note that the graph of $y = x^2 - 1$ is shifted 1 unit downward from the graph of $y = x^2$ and the vertex of the graph of $y = x^2 - 1$ is now at the point $(0, -1)$.



(a) (b)

Figure 11. Drawing $y = x^2$ and $y = x^2 - 1$ on the graphing calculator.

The above discussion leads to the following property.

Property 13. If $c > 0$, the graph of $g(x) = x^2 - c$ is shifted c units downward from the graph of $f(x) = x^2$.

The Axis of Symmetry

In **Figure 1**, the graph of $y = x^2$ is symmetric with respect to the y-axis. One half of the parabola is a mirror image of the other with respect to the y-axis. We say the y-axis is acting as the *axis of symmetry*.

If the parabola is reflected across the x-axis, as in **Figure 6**, the axis of symmetry doesn't change. The graph is still symmetric with respect to the y-axis. Similar comments are in order for scalings and vertical translations. However, if the graph of $y = x^2$ is shifted right or left, then the axis of symmetry will change.

I **Example 14.** *Sketch the graph of* $y = -(x + 2)^2 + 3$.

Although not required, this example is much simpler if you perform reflections before translations.

Tip 15. *If at all possible, perform scalings and reflections before translations.*

In the series shown in **Figure 12**, we first perform a reflection, then a horizontal translation, followed by a vertical translation.

- In **Figure 12**(a), the graph of $y = -x^2$ is a reflection of the graph of $y = x^2$ across the x-axis and opens downward. Note that the vertex is still at the origin.
- In **Figure 12**(b), we've replaced x with $x+2$ in the equation $y = -x^2$ to obtain the equation $y = -(x+2)^2$. The effect is to shift the graph of $y = -x^2$ in **Figure 12**(a) 2 units to the left to obtain the graph of $y = -(x+2)^2$ in **Figure 12**(b). Note that the vertex is now at the point $(-2, 0)$.
- In **Figure 12**(c), we've added 3 to the equation $y = -(x+2)^2$ to obtain the equation $y = -(x+2)^2 + 3$. The effect is to shift the graph of $y = -(x+2)^2$ in **Figure 12**(b)

upward 3 units to obtain the graph of $y = -(x+2)^2 + 3$ in **Figure 12**(c). Note that the vertex is now at the point $(-2, 3)$.

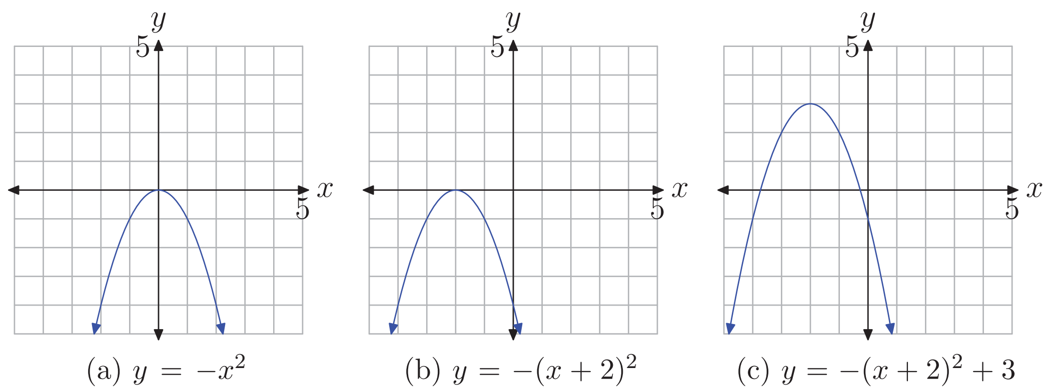


Figure 12. Finding the graph of $y = -(x+2)^2 + 3$ through a series of transformations.

In practice, we can proceed more quickly. Analyze the equation $y = -(x+2)^2 + 3$. The minus sign tells us that the parabola "opens downward." The presence of $x + 2$ indicates a shift of 2 units to the left. Finally, adding the 3 will shift the graph 3 units upward. Thus, we have a parabola that "opens downward" with vertex at $(-2, 3)$. This is shown in **Figure 13**.

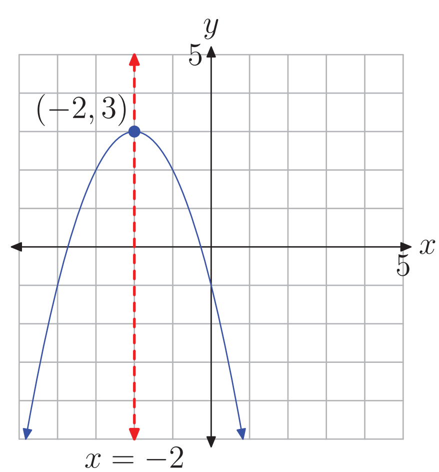


Figure 13. The axis of symmetry passes through the vertex.

The axis of symmetry passes through the vertex $(-2, 3)$ in **Figure 13** and has equation $x = -2$. Note that the right half of the parabola is a mirror image of its left half across this axis of symmetry. We can use the axis of symmetry to gain an accurate plot of the parabola with minimal plotting of points.

Guidelines for Using the Axis of Symmetry.

- Start by plotting the vertex and axis of symmetry as shown in **Figure 14**(a).
- Next, compute two points on either side of the axis of symmetry. We choose $x = -1$ and $x = 0$ and compute the corresponding y-values using the equation $y = -(x+2)^2 + 3$.

x	$y = -(x+2)^2 + 3$
-1	2
0	-1

  Plot the points from the table, as shown in **Figure 14**(b).
- Finally, plot the mirror images of these points across the axis of symmetry, as shown in **Figure 14**(c).

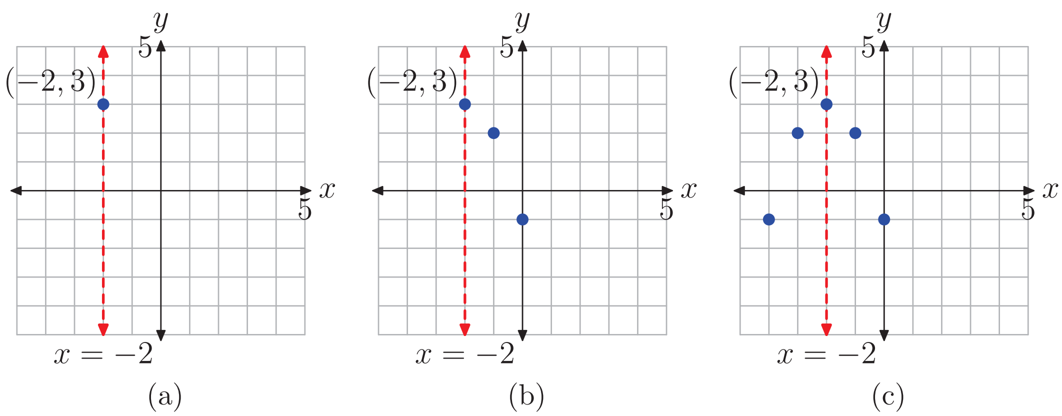


Figure 14. Using the axis of symmetry to establish accuracy.

The image in **Figure 14**(c) clearly contains enough information to complete the graph of the parabola having equation $y = -(x+2)^2 + 3$ in **Figure 15**.

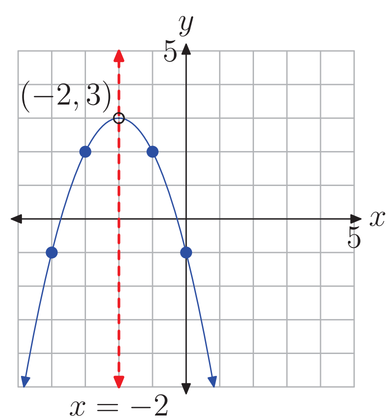


Figure 15. An accurate plot of $y = -(x+2)^2 + 3$.

Let's summarize what we've seen thus far.

Summary 16. *The form of the quadratic function*

$$f(x) = a(x - h)^2 + k$$

is called **vertex form**. *The graph of this quadratic function is a* **parabola**.

1. *The graph of the parabola opens* upward *if $a > 0$,* downward *if $a < 0$.*
2. *If the magnitude of a is larger than 1, then the graph of the parabola is* stretched *by a factor of a. If the magnitude of a is smaller than 1, then the graph of the parabola is* compressed *by a factor of $1/a$.*
3. *The parabola is translated h units to the* right *if $h > 0$, and h units to the* left *if $h < 0$.*
4. *The parabola is translated k units* upward *if $k > 0$, and k units* downward *if $k < 0$.*
5. *The coordinates of the* vertex *are (h, k).*
6. *The* axis of symmetry *is a vertical line through the vertex whose equation is $x = h$.*

Let's look at one final example.

I **Example 17.** *Use the technique of* **Example 14** *to sketch the graph of $f(x) = 2(x - 2)^2 - 3$.*

Compare $f(x) = 2(x - 2)^2 - 3$ with $f(x) = a(x - h)^2 + k$ and note that $a = 2$. Hence, the parabola has been "stretched" by a factor of 2 and opens upward. The presence of $x - 2$ indicates a shift of 2 units to the right; and subtracting 3 shifts the parabola 3 units downward. Therefore, the vertex will be located at the point $(2, -3)$ and the axis of symmetry will be the vertical line having equation $x = 2$. This is shown in **Figure 16**(a).

Note. Some prefer a more strict comparison of $f(x) = 2(x - 2)^2 - 3$ with the general vertex form $f(x) = a(x - h)^2 + k$, yielding $a = 2$, $h = 2$, and $k = -3$. This immediately identifies the vertex at (h, k), or $(2, -3)$.

Next, evaluate the function $f(x) = 2(x - 2)^2 - 3$ at two points lying to the right of the axis of symmetry (or to the left, if you prefer). Because the axis of symmetry is the vertical line $x = 2$, we choose to evaluate the function at $x = 3$ and 4.

$$f(3) = 2(3 - 2)^2 - 3 = -1$$
$$f(4) = 2(4 - 2)^2 - 3 = 5$$

This gives us two points to the right of the axis of symmetry, $(3, -1)$ and $(4, 5)$, which we plot in **Figure 16**(b).

Finally, we plot the mirror images of $(3, -1)$ and $(4, 5)$ across the axis of symmetry, which gives us the points $(1, -1)$ and $(0, 5)$, respectively. These are plotted in **Figure 16**(c). We then draw the parabola through these points.

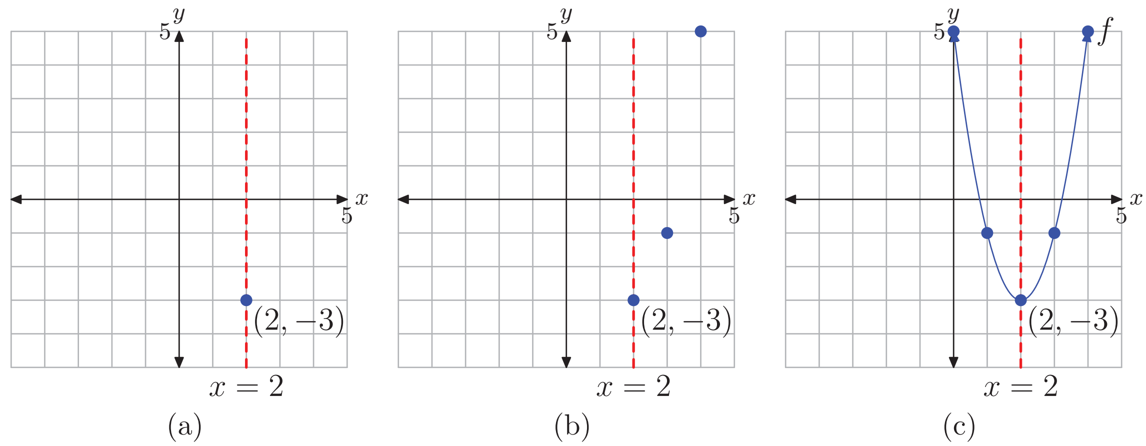


Figure 16. Creating the graph of $f(x) = 2(x-2)^2 - 3$.

Let's finish by describing the domain and range of the function defined by the rule $f(x) = 2(x-2)^2 - 3$. If you use the intuitive notion that the domain is the set of "permissible x-values," then one can substitute any number one wants into the equation $f(x) = 2(x-2)^2 - 3$. Therefore, the domain is all real numbers, which we can write as follows: Domain $= \mathbb{R}$ or Domain $= (-\infty, \infty)$.

You can also project each point on the graph of $f(x) = 2(x-2)^2 - 3$ onto the x-axis, as shown in **Figure 17**(a). If you do this, then the entire axis will "lie in shadow," so once again, the domain is all real numbers.

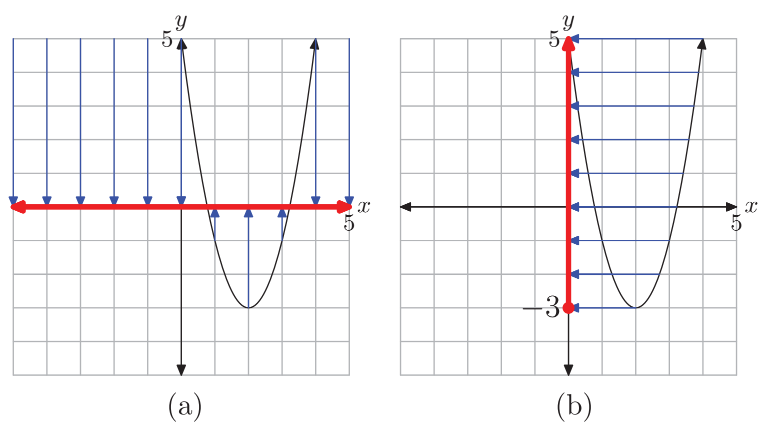


Figure 17. Projecting to find
(a) the domain and (b) the range.

To determine the range of the function $f(x) = 2(x-2)^2 - 3$, project each point on the graph of f onto the y-axis, as shown in **Figure 17**(b). On the y-axis, all points greater than or equal to -3 "lie in shadow," so the range is described with Range $= \{y : y \geq -3\} = [-3, \infty)$.

The following summarizes how one finds the domain and range of a quadratic function that is in vertex form.

Summary 18. *The domain of the quadratic function*

$$f(x) = a(x-h)^2 + k,$$

regardless of the values of the parameters a, h, and k, is the set of all real numbers, easily described with $\mathbb{R}$ or $(-\infty, \infty)$. On the other hand, the range depends upon the values of a and k.

- *If $a > 0$, then the parabola opens upward and has vertex at (h, k). Consequently, the range will be*

$$[k, \infty) = \{y : y \geq k\}.$$

- *If $a < 0$, then the parabola opens downward and has vertex at (h, k). Consequently, the range will be*

$$(-\infty, k] = \{y : y \leq k\}.$$

2.1 Exercises

In **Exercises 1-6**, sketch the image of your calculator screen on your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Label each graph with its equation. *Remember to use a ruler to draw all lines, including axes.*

1. Use your graphing calculator to sketch the graphs of $f(x) = x^2$, $g(x) = 2x^2$, and $h(x) = 4x^2$ on one screen. Write a short sentence explaining what you learned in this exercise.

2. Use your graphing calculator to sketch the graphs of $f(x) = -x^2$, $g(x) = -2x^2$, and $h(x) = -4x^2$ on one screen. Write a short sentence explaining what you learned in this exercise.

3. Use your graphing calculator to sketch the graphs of $f(x) = x^2$, $g(x) = (x - 2)^2$, and $h(x) = (x - 4)^2$ on one screen. Write a short sentence explaining what you learned in this exercise.

4. Use your graphing calculator to sketch the graphs of $f(x) = x^2$, $g(x) = (x + 2)^2$, and $h(x) = (x + 4)^2$ on one screen. Write a short sentence explaining what you learned in this exercise.

5. Use your graphing calculator to sketch the graphs of $f(x) = x^2$, $g(x) = x^2 + 2$, and $h(x) = x^2 + 4$ on one screen. Write a short sentence explaining what you learned in this exercise.

6. Use your graphing calculator to sketch the graphs of $f(x) = x^2$, $g(x) = x^2 - 2$, and $h(x) = x^2 - 4$ on one screen. Write a short sentence explaining what you learned in this exercise.

In **Exercises 7-14**, write down the given quadratic function on your homework paper, then state the coordinates of the vertex.

7. $f(x) = -5(x - 4)^2 - 5$

8. $f(x) = 5(x + 3)^2 - 7$

9. $f(x) = 3(x + 1)^2$

10. $f(x) = \frac{7}{5}\left(x + \frac{5}{9}\right)^2 - \frac{3}{4}$

11. $f(x) = -7(x - 4)^2 + 6$

12. $f(x) = -\frac{1}{2}\left(x - \frac{8}{9}\right)^2 + \frac{2}{9}$

13. $f(x) = \frac{1}{6}\left(x + \frac{7}{3}\right)^2 + \frac{3}{8}$

14. $f(x) = -\frac{3}{2}\left(x + \frac{1}{2}\right)^2 - \frac{8}{9}$

In **Exercises 15-22**, state the equation of the axis of symmetry of the graph of the given quadratic function.

15. $f(x) = -7(x - 3)^2 + 1$

16. $f(x) = -6(x + 8)^2 + 1$

17. $f(x) = -\dfrac{7}{8}\left(x + \dfrac{1}{4}\right)^2 + \dfrac{2}{3}$

18. $f(x) = -\dfrac{1}{2}\left(x - \dfrac{3}{8}\right)^2 - \dfrac{5}{7}$

19. $f(x) = -\dfrac{2}{9}\left(x + \dfrac{2}{3}\right)^2 - \dfrac{4}{5}$

20. $f(x) = -7(x+3)^2 + 9$

21. $f(x) = -\dfrac{8}{7}\left(x + \dfrac{2}{9}\right)^2 + \dfrac{6}{5}$

22. $f(x) = 3(x+3)^2 + 6$

In **Exercises 23-36**, perform each of the following tasks for the given quadratic function.

i. Set up a coordinate system on graph paper. Label and scale each axis.
ii. Plot the vertex of the parabola and label it with its coordinates.
iii. Draw the axis of symmetry and label it with its equation.
iv. Set up a table near your coordinate system that contains exact coordinates of two points on either side of the axis of symmetry. Plot them on your coordinate system and their "mirror images" across the axis of symmetry.
v. Sketch the parabola and label it with its equation.
vi. Use interval notation to describe both the domain and range of the quadratic function.

23. $f(x) = (x+2)^2 - 3$

24. $f(x) = (x-3)^2 - 4$

25. $f(x) = -(x-2)^2 + 5$

26. $f(x) = -(x+4)^2 + 4$

27. $f(x) = (x-3)^2$

28. $f(x) = -(x+2)^2$

29. $f(x) = -x^2 + 7$

30. $f(x) = -x^2 + 7$

31. $f(x) = 2(x-1)^2 - 6$

32. $f(x) = -2(x+1)^2 + 5$

33. $f(x) = -\frac{1}{2}(x+1)^2 + 5$

34. $f(x) = \frac{1}{2}(x-3)^2 - 6$

35. $f(x) = 2(x-5/2)^2 - 15/2$

36. $f(x) = -3(x+7/2)^2 + 15/4$

In **Exercises 37-44**, write the given quadratic function on your homework paper, then use set-builder and interval notation to describe the domain and the range of the function.

37. $f(x) = 7(x+6)^2 - 6$

38. $f(x) = 8(x+1)^2 + 7$

39. $f(x) = -3(x+4)^2 - 7$

40. $f(x) = -6(x-7)^2 + 9$

41. $f(x) = -7(x+5)^2 - 7$

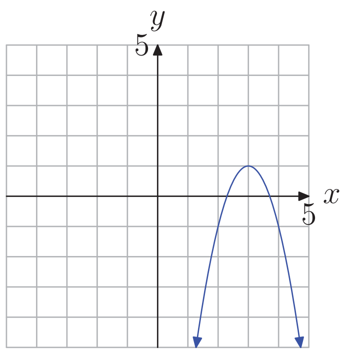
42. $f(x) = 8(x-4)^2 + 3$

43. $f(x) = -4(x-1)^2 + 2$

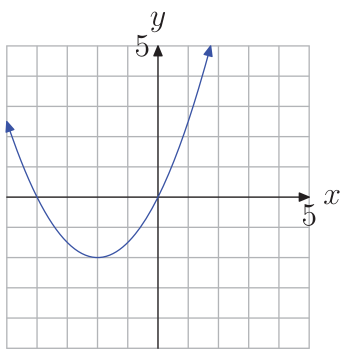
44. $f(x) = 7(x-2)^2 - 3$

In **Exercises 45-52**, using the given value of a, find the specific quadratic function of the form $f(x) = a(x-h)^2 + k$ that has the graph shown. Note: h and k are integers. Check your solution with your graphing calculator.

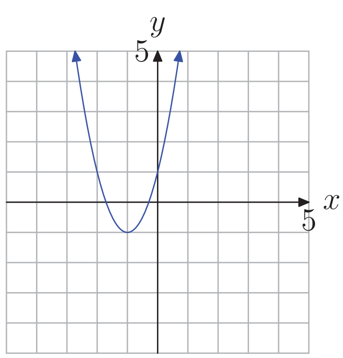
45. $a = -2$



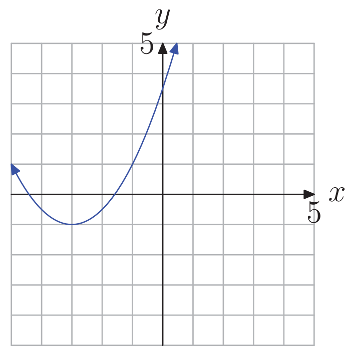
46. $a = 0.5$



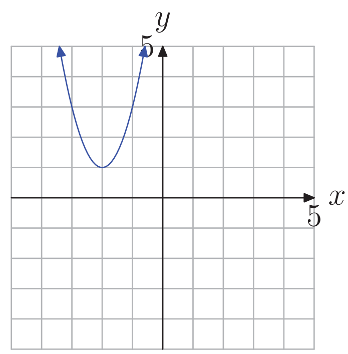
47. $a = 2$



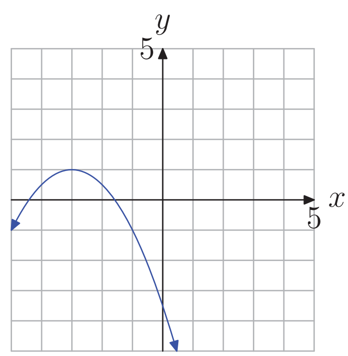
48. $a = 0.5$



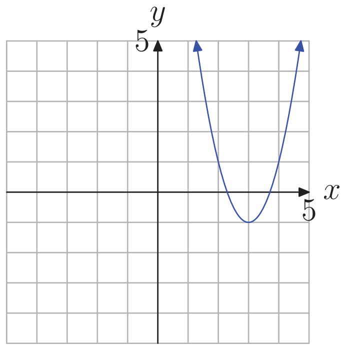
49. $a = 2$



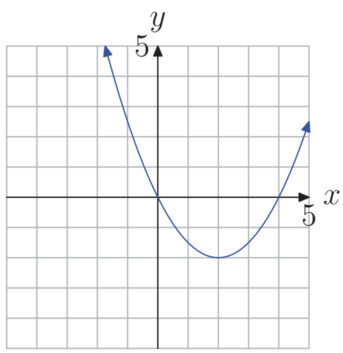
50. $a = -0.5$



51. $a = 2$

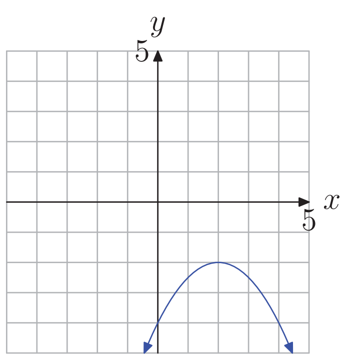


52. $a = 0.5$

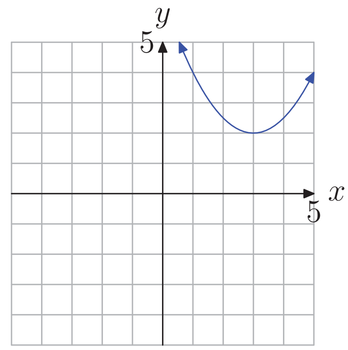


In **Exercises 53-54**, use the graph to determine the range of the function $f(x) = ax^2 + bx + c$. The arrows on the graph are meant to indicate that the graph continues indefinitely in the continuing pattern and direction of each arrow. Describe your solution using interval notation.

53.

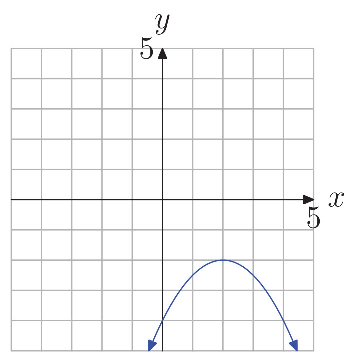


54.

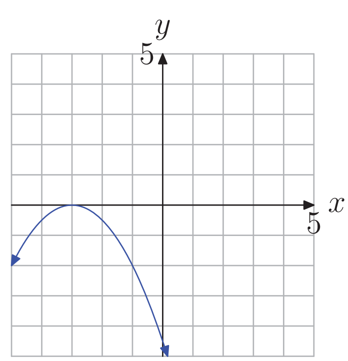


In **Exercises 55-56**, use the graph to determine the domain of the function $f(x) = ax^2 + bx + c$. The arrows on the graph are meant to indicate that the graph continues indefinitely in the continuing pattern and direction of each arrow. Use interval notation to describe your solution.

55.

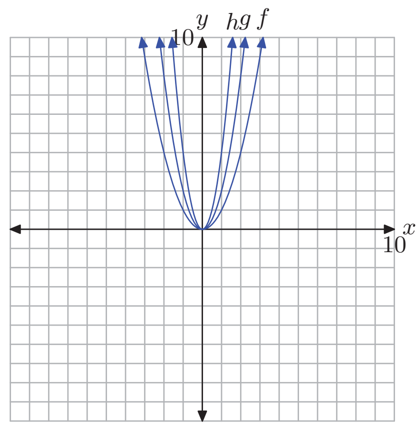


56.

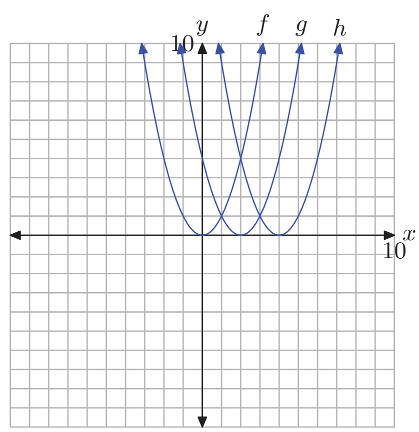


2.1 Answers

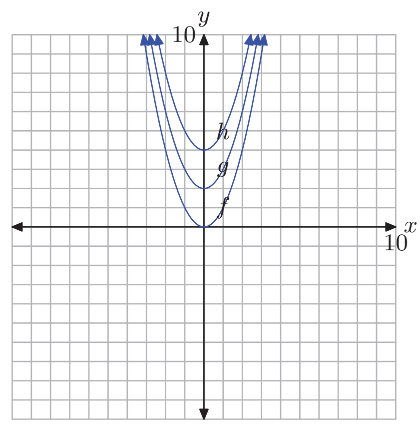
1. Multiplying by 2 scales vertically by a factor of 2. Multiplying by 4 scales vertically by a factor of 4.



3. The graph of $g(x) = (x-2)^2$ is shifted 2 units to the right of $f(x) = x^2$. The graph of $h(x) = (x-4)^2$ is shifted 4 units to the right of $f(x) = x^2$.



5. The graph of $g(x) = x^2 + 2$ is shifted 2 units to the upward from the graph of $f(x) = x^2$. The graph of $h(x) = x^2 + 4$ is shifted 4 units upward from the graph of $f(x) = x^2$.



7. $(4, -5)$

9. $(-1, 0)$

11. $(4, 6)$

13. $\left(-\frac{7}{3}, \frac{3}{8}\right)$

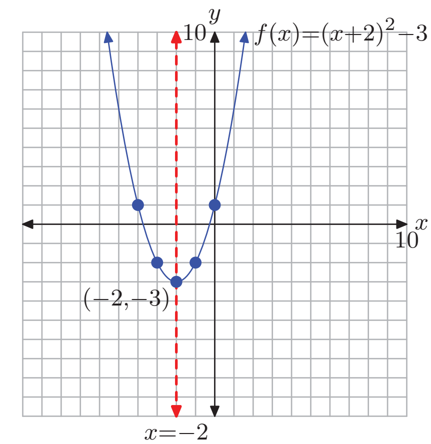
15. $x = 3$

17. $x = -\frac{1}{4}$

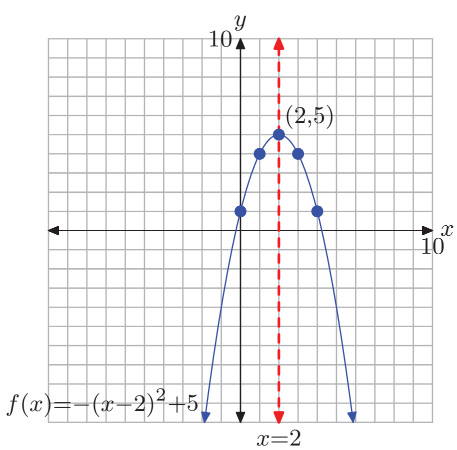
19. $x = -\frac{2}{3}$

21. $x = -\frac{2}{9}$

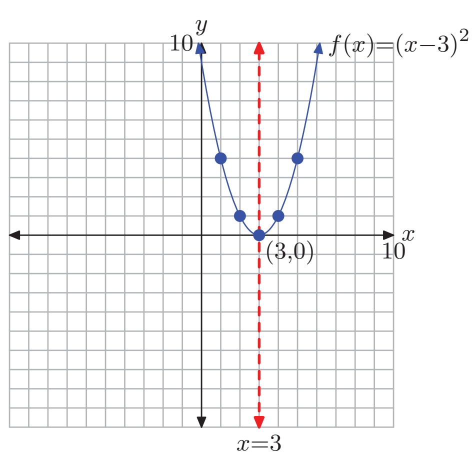
23. Domain= $(-\infty, \infty)$; Range= $[-3, \infty)$



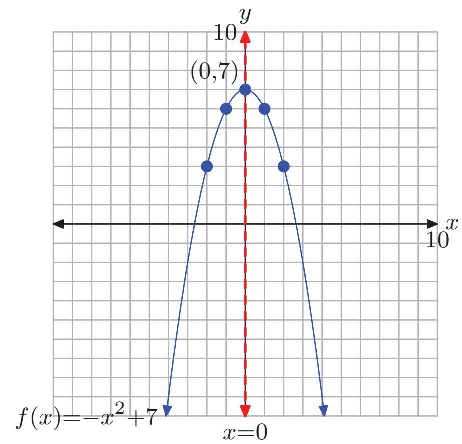
25. Domain= $(-\infty, \infty)$; Range= $(-\infty, 5]$



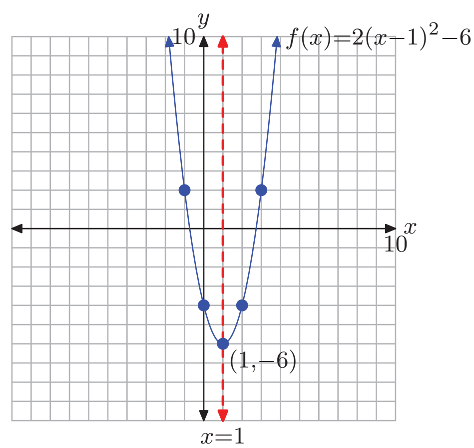
27. Domain= $(-\infty, \infty)$; Range= $[0, \infty)$



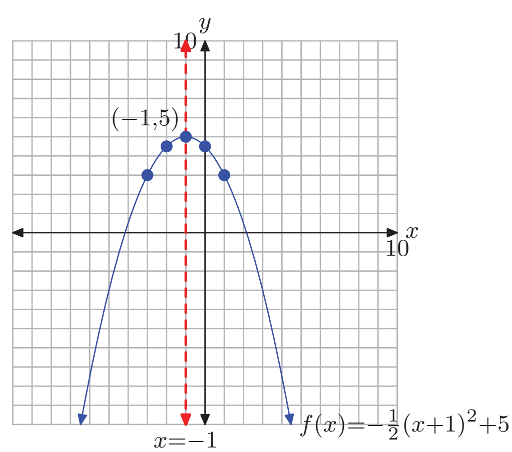
29. Domain= $(-\infty, \infty)$; Range= $(-\infty, 7]$



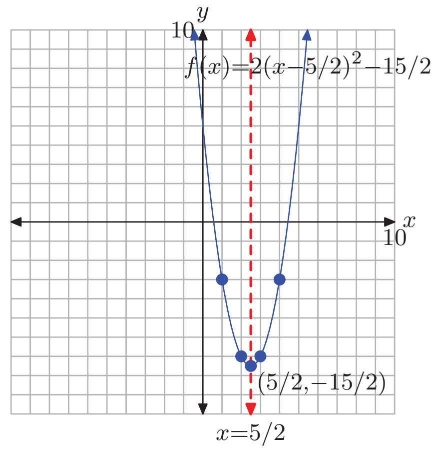
31. Domain= $(-\infty, \infty)$; Range= $[-6, \infty)$



33. Domain= $(-\infty, \infty)$; Range= $(-\infty, 5]$



35. Domain= $(-\infty, \infty)$; Range= $[-15/2, \infty)$



37. Domain= $(-\infty, \infty)$; Range= $[-6, \infty) = \{y : y \geq -6\}$

39. Domain= $(-\infty, \infty)$; Range= $(-\infty, -7] = \{y : y \leq -7\}$

41. Domain= $(-\infty, \infty)$; Range= $(-\infty, -7] = \{y : y \leq -7\}$

43. Domain= $(-\infty, \infty)$; Range= $(-\infty, 2] = \{y : y \leq 2\}$

45. $f(x) = -2(x-3)^2 + 1$

47. $f(x) = 2(x+1)^2 - 1$

49. $f(x) = 2(x+2)^2 + 1$

51. $f(x) = 2(x-3)^2 - 1$

53. $(-\infty, -2]$

55. $(-\infty, \infty)$

2.2 Vertex Form

In the previous section, you learned that it is a simple task to sketch the graph of a quadratic function if it is presented in *vertex form*

$$f(x) = a(x-h)^2 + k. \tag{1}$$

The goal of the current section is to start with the most general form of the quadratic function, namely

$$f(x) = ax^2 + bx + c, \tag{2}$$

and manipulate the equation into *vertex form*. Once you have your quadratic function in vertex form, the technique of the previous section should allow you to construct the graph of the quadratic function.

However, before we turn our attention to the task of converting the general quadratic into vertex form, we need to review the necessary algebraic fundamentals. Let's begin with a review of an important algebraic shortcut called *squaring a binomial*.

Squaring a Binomial

A *monomial* is a single algebraic term, usually constructed as a product of a number (called a *coefficient*) and one or more variables raised to nonnegative integral powers, such as $-3x^2$ or $14y^3z^5$. The key phrase here is "single term." A *binomial* is an algebraic sum or difference of two monomials (or terms), such as $x + 2y$ or $3ab^2 - 2c^3$. The key phrase here is "two terms."

To "square a binomial," start with an arbitrary binomial, such as $a+b$, then multiply it by itself to produce its square $(a+b)(a+b)$, or, more compactly, $(a+b)^2$. We can use the distributive property to expand the square of the binomial $a+b$.

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a(a+b) + b(a+b)\\ &= a^2 + ab + ba + b^2\end{aligned}$$

Because $ab = ba$, we can add the two middle terms to arrive at the following property.

Property 3. The square of the binomial $a+b$ is expanded as follows.

$$(a+b)^2 = a^2 + 2ab + b^2 \tag{4}$$

[3] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

I **Example 5.** *Expand* $(x+4)^2$.

We could proceed as follows.

$$\begin{aligned}(x+4)^2 &= (x+4)(x+4)\\ &= x(x+4)+4(x+4)\\ &= x^2+4x+4x+16\\ &= x^2+8x+16\end{aligned}$$

Although correct, this technique will not help us with our upcoming task. What we need to do is follow the algorithm suggested by **Property 3**.

Algorithm for Squaring a Binomial. To square the binomial $a+b$, proceed as follows:

1. Square the first term to get a^2.
2. Multiply the first and second terms together, and then multiply the result by two to get $2ab$.
3. Square the second term to get b^2.

Thus, to expand $(x+4)^2$, we should proceed as follows.

1. Square the first term to get x^2
2. Multiply the first and second terms together and then multiply by two to get $8x$.
3. Square the second term to get 16.

Proceeding in this manner allows us to perform the expansion mentally and simply write down the solution.

$$(x+4)^2 = x^2+2(x)(4)+4^2 = x^2+8x+16$$

Here are a few more examples. In each, we've written an extra step to help clarify the procedure. In practice, you should simply write down the solution without any intermediate steps.

$$\begin{aligned}(x+3)^2 &= x^2+2(x)(3)+3^2 = x^2+6x+9\\ (x-5)^2 &= x^2+2(x)(-5)+(-5)^2 = x^2-10x+25\\ \left(x-\tfrac{1}{2}\right)^2 &= x^2+2(x)\left(-\tfrac{1}{2}\right)+\left(-\tfrac{1}{2}\right)^2 = x^2-x+\tfrac{1}{4}\end{aligned}$$

It is imperative that you master this shortcut before moving on to the rest of the material in this section.

Perfect Square Trinomials

Once you've mastered squaring a binomial, as in

$$(a+b)^2 = a^2 + 2ab + b^2, \tag{6}$$

it's a simple matter to identify and factor trinomials (three terms) having the form $a^2 + 2ab + b^2$. You simply "undo" the multiplication. Whenever you spot a trinomial whose first and third terms are perfect squares, you should suspect that it factors as follows.

$$a^2 + 2ab + b^2 = (a+b)^2 \tag{7}$$

A trinomial that factors according to this rule or pattern is called a *perfect square trinomial*.

For example, the first and last terms of the following trinomial are perfect squares.

$$x^2 + 16x + 64$$

The square roots of the first and last terms are x and 8, respectively. Hence, it makes sense to try the following.

$$x^2 + 16x + 64 = (x+8)^2$$

It is important that you check your result using multiplication. So, following the three-step algorithm for squaring a binomial:

1. Square x to get x^2.
2. Multiply x and 8 to get $8x$, then multiply this result by 2 to get $16x$.
3. Square 8 to get 64.

Hence, $x^2 + 16x + 64$ is a perfect square trinomial and factors as $(x+8)^2$.

As another example, consider $x^2 - 10x + 25$. The square roots of the first and last terms are x and 5, respectively. Hence, it makes sense to try

$$x^2 - 10x + 25 = (x-5)^2.$$

Again, you should check this result. Note especially that twice the product of x and -5 equals the middle term on the left, namely, $-10x$.

Completing the Square

If a quadratic function is given in vertex form, it is a simple matter to sketch the parabola represented by the equation. For example, consider the quadratic function

$$f(x) = (x+2)^2 + 3,$$

which is in vertex form. The graph of this equation is a parabola that opens upward. It is translated 2 units to the left and 3 units upward. This is the advantage of vertex form. The transformations required to draw the graph of the function are easy to spot when the equation is written in vertex form.

It's a simple matter to transform the equation $f(x) = (x+2)^2 + 3$ into the general form of a quadratic function, $f(x) = ax^2 + bx + c$. We simply use the three-step algorithm to square the binomial; then we combine like terms.

$$\begin{aligned} f(x) &= (x+2)^2 + 3 \\ f(x) &= x^2 + 4x + 4 + 3 \\ f(x) &= x^2 + 4x + 7 \end{aligned}$$

Note, however, that the result of this manipulation, $f(x) = x^2 + 4x + 7$, is not as useful as vertex form, as it is difficult to identify the transformations required to draw the parabola represented by the equation $f(x) = x^2 + 4x + 7$.

It's really the *reverse* of the manipulation above that is needed. If we are presented with an equation in the form $f(x) = ax^2 + bx + c$, such as $f(x) = x^2 + 4x + 7$, then an algebraic method is needed to convert this equation to vertex form $f(x) = a(x-h)^2 + k$; or in this case, back to its original vertex form $f(x) = (x+2)^2 + 3$.

The procedure we seek is called *completing the square*. The name is derived from the fact that we need to "complete" the trinomial on the right side of $y = x^2 + 4x + 7$ so that it becomes a perfect square trinomial.

Algorithm for Completing the Square The procedure for *completing the square* involves three key steps.

1. Take half of the coefficient of x and square the result.
2. Add and subtract the quantity from step one so that the right-hand side of the equation does not change.
3. Factor the resulting perfect square trinomial and combine constant terms.

Let's follow this procedure and place $f(x) = x^2 + 4x + 7$ in vertex form.

1. Take half of the coefficient of x. Thus, $(1/2)(4) = 2$. Square this result. Thus, $2^2 = 4$.

2. Add and subtract 4 on the right side of the equation $f(x) = x^2 + 4x + 7$.

$$f(x) = x^2 + 4x + 4 - 4 + 7$$

3. Group the first three terms on the right-hand side. These form a perfect square trinomial.

$$f(x) = (x^2 + 4x + 4) - 4 + 7$$

Now factor the perfect square trinomial and combine the constants at the end to get

$$f(x) = (x+2)^2 + 3.$$

That's it, we're done! We've returned the general quadratic $f(x) = x^2 + 4x + 7$ back to vertex form $f(x) = (x + 2)^2 + 3$.

Let's try that once more.

I **Example 8.** *Place the quadratic function $f(x) = x^2 - 8x - 9$ in vertex form.*

We follow the three-step algorithm for completing the square.

1. Take half of the coefficient of x and square: i.e.,

$$[(1/2)(-8)]^2 = [-4]^2 = 16.$$

2. Add and subtract this amount to the right-hand side of the function.

$$f(x) = x^2 - 8x + 16 - 16 - 9$$

3. Group the first three terms on the right-hand side. These form a perfect square trinomial.

$$f(x) = (x^2 - 8x + 16) - 16 - 9$$

Factor the perfect square trinomial and combine the coefficients at the end.

$$f(x) = (x - 4)^2 - 25$$

Now, let's see how we can use the technique of completing the square to help in drawing the graphs of general quadratic functions.

Working with $f(x) = x^2 + bx + c$

The examples in this section will have the form $f(x) = x^2 + bx + c$. Note that the coefficient of x^2 is 1. In the next section, we will work with a harder form, $f(x) = ax^2 + bx + c$, where $a \neq 1$.

I **Example 9.** *Complete the square to place $f(x) = x^2 + 6x + 2$ in vertex form and sketch its graph.*

First, take half of the coefficient of x and square; i.e., $[(1/2)(6)]^2 = 9$. On the right side of the equation, add and subtract this amount so as to not change the equation.

$$f(x) = x^2 + 6x + 9 - 9 + 2$$

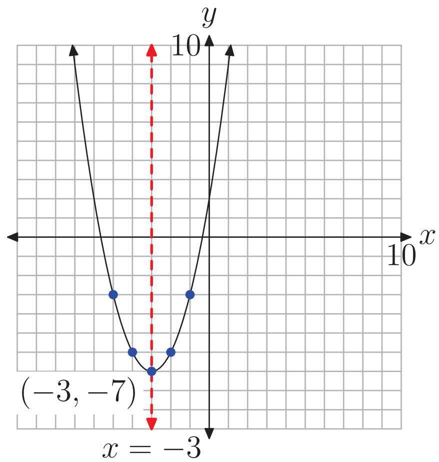
Group the first three terms on the right-hand side.

$$f(x) = (x^2 + 6x + 9) - 9 + 2$$

The first three terms on the right-hand side form a perfect square trinomial that is easily factored. Also, combine the constants at the end.

$$f(x) = (x + 3)^2 - 7$$

This is a parabola that opens upward. We need to shift the parabola 3 units to the left and then 7 units downward, placing the vertex at $(-3, -7)$ as shown in **Figure 1**(a). The axis of symmetry is the vertical line $x = -3$. The table in **Figure 1**(b) calculates two points to the right of the axis of symmetry, and mirror points on the left of the axis of symmetry make for an accurate plot of the parabola.



(a)

x	$y = (x+3)^2 - 7$
-2	-6
-1	-3

(b)

Figure 1. Plotting the graph of the quadratic function $f(x) = (x+3)^2 - 7$.

Let's look at another example.

I **Example 10.** *Complete the square to place* $f(x) = x^2 - 8x + 21$ *in vertex form and sketch its graph.*

First, take half of the coefficient of x and square; i.e., $[(1/2)(-8)]^2 = 16$. On the right side of the equation, add and subtract this amount so as to not change the equation.

$$f(x) = x^2 - 8x + 16 - 16 + 21$$

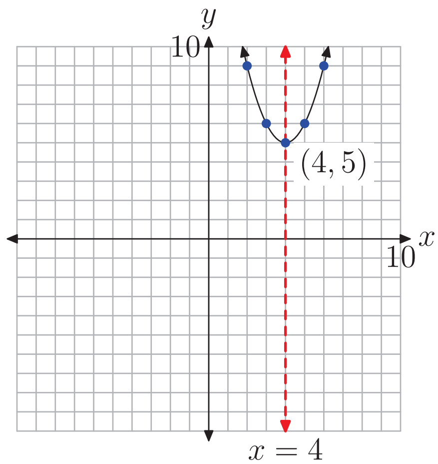
Group the first three terms on the right-hand side of the equation.

$$f(x) = (x^2 - 8x + 16) - 16 + 21$$

The first three terms form a perfect square trinomial that is easily factored. Also, combine constants at the end.

$$f(x) = (x-4)^2 + 5$$

This is a parabola that opens upward. We need to shift the parabola 4 units to the right and then 5 units upward, placing the vertex at $(4, 5)$, as shown in **Figure 2**(a). The table in **Figure 2**(b) calculates two points to the right of the axis of symmetry, and mirror points on the left of the axis of symmetry make for an accurate plot of the parabola.



(a)

x	$y = (x-4)^2 + 5$
5	6
6	9

(b)

Figure 2. Plotting the graph of the quadratic function $f(x) = (x-4)^2 + 5$.

Working with $f(x) = ax^2 + bx + c$

In the last two examples, the coefficient of x^2 was 1. In this , we will learn how to complete the square when the coefficient of x^2 is some number other than 1.

I **Example 11.** *Complete the square to place $f(x) = 2x^2 + 4x - 4$ in vertex form and sketch its graph.*

In the last two examples, we gained some measure of success when the coefficient of x^2 was a 1. We were just getting comfortable with that situation and we'd like to continue to be comfortable, so let's start by factoring a 2 from each term on the right-hand side of the equation.

$$f(x) = 2\big[x^2 + 2x - 2\big]$$

If we ignore the factor of 2 out front, the coefficient of x^2 in the trinomial expression inside the parentheses is a 1. Ah, familiar ground! We will proceed as we did before, but we will carry the factor of 2 outside the parentheses in each step. Start by taking half of the coefficient of x and squaring the result; i.e., $[(1/2)(2)]^2 = 1$.

Add and subtract this amount inside the parentheses so as to not change the equation.

$$f(x) = 2\big[x^2 + 2x + 1 - 1 - 2\big]$$

Group the first three terms inside the parentheses and combine constants.

$$f(x) = 2\big[(x^2 + 2x + 1) - 3\big]$$

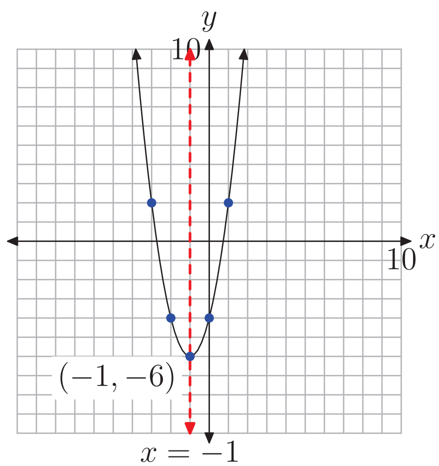
The grouped terms inside the parentheses form a perfect square trinomial that is easily factored.

$$f(x) = 2\big[(x+1)^2 - 3\big]$$

Finally, redistribute the 2.

$$f(x) = 2(x+1)^2 - 6$$

This is a parabola that opens upward. In addition, it is stretched by a factor of 2, so it will be somewhat narrower than our previous examples. The parabola is also shifted 1 unit to the left, then 6 units downward, placing the vertex at $(-1,-6)$, as shown in **Figure 3**(a). The table in **Figure 3**(b) calculates two points to the right of the axis of symmetry, and mirror points on the left of the axis of symmetry make for an accurate plot of the parabola.



(a)

x	$y = 2(x+1)^2 - 6$
0	-4
1	2

(a)

Figure 3. Plotting the graph of the quadratic function $f(x) = 2x^2 + 4x - 4$.

Let's look at an example where the coefficient of x^2 is negative.

I **Example 12.** *Complete the square to place $f(x) = -x^2 + 6x - 2$ in vertex form and sketch its graph.*

In the last example, we factored out the coefficient of x^2. This left us with a trinomial having leading coefficient[4] 1, which enabled us to proceed much as we did before: halve the middle coefficient and square, add and subtract this amount, factor the resulting perfect square trinomial. Since we were successful with this technique in the last example, let's begin again by factoring out the leading coefficient, in this case a -1.

[4] The leading coefficient of a quadratic function is the coefficient of x^2. That is, if $f(x) = ax^2 + bx + c$, then the leading coefficient is "a." We'll have more to say about the leading coefficient in Chapter 6.

$$f(x) = -1\left[x^2 - 6x + 2\right]$$

If we ignore the factor of -1 out front, the coefficient of x^2 in the trinomial expression inside the parentheses is a 1. Again, familiar ground! We will proceed as we did before, but we will carry the factor of -1 outside the parentheses in each step. Start by taking half of the coefficient of x and squaring the result; i.e., $[(1/2)(-6)]^2 = 9$.

Add and subtract this amount inside the parentheses so as to not change the equation.

$$f(x) = -1\left[x^2 - 6x + 9 - 9 + 2\right]$$

Group the first three terms inside the parentheses and combine constants.

$$f(x) = -1\left[(x^2 - 6x + 9) - 7\right]$$

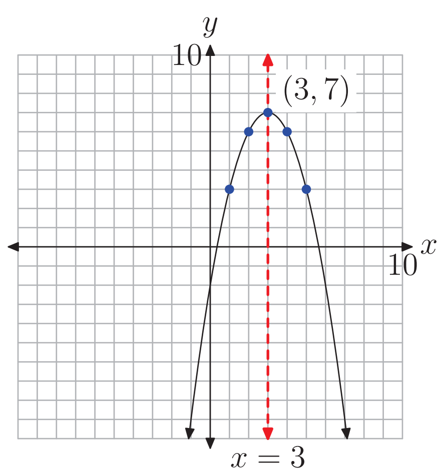
The grouped terms inside the parentheses form a perfect square trinomial that is easily factored.

$$f(x) = -1\left[(x-3)^2 - 7\right]$$

Finally, redistribute the -1.

$$f(x) = -(x-3)^2 + 7$$

This is a parabola that opens downward. The parabola is also shifted 3 units to the right, then 7 units upward, placing the vertex at $(3, 7)$, as shown in **Figure 4**(a). The table in **Figure 4**(b) calculates two points to the right of the axis of symmetry, and mirror points on the left of the axis of symmetry make for an accurate plot of the parabola.



(a)

x	$y = -(x-3)^2 + 7$
4	6
5	3

(b)

Figure 4. Plotting the graph of the quadratic function $f(x) = -(x-3)^2 + 7$.

Let's try one more example.

I **Example 13.** *Complete the square to place $f(x) = 3x^2 + 4x - 8$ in vertex form and sketch its graph.*

Let's begin again by factoring out the leading coefficient, in this case a 3.

$$f(x) = 3\left[x^2 + \tfrac{4}{3}x - \tfrac{8}{3}\right]$$

Fractions add a degree of difficulty, but, if you follow the same routine as in the previous examples, you should be able to get the needed result. Take half of the coefficient of x and square the result; i.e., $[(1/2)(4/3)]^2 = [2/3]^2 = 4/9$.

Add and subtract this amount inside the parentheses so as to not change the equation.

$$f(x) = 3\left[x^2 + \tfrac{4}{3}x + \tfrac{4}{9} - \tfrac{4}{9} - \tfrac{8}{3}\right]$$

Group the first three terms inside the parentheses. You'll need a common denominator to combine constants.

$$f(x) = 3\left[\left(x^2 + \tfrac{4}{3}x + \tfrac{4}{9}\right) - \tfrac{4}{9} - \tfrac{24}{9}\right]$$

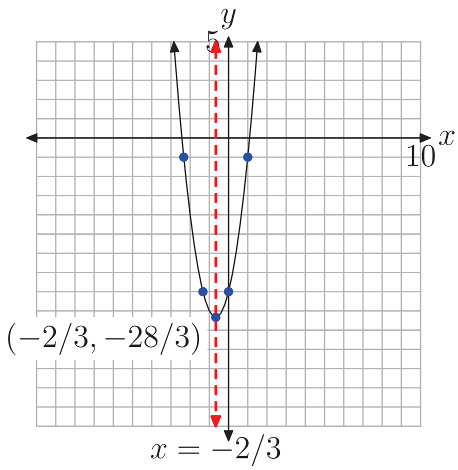
The grouped terms inside the parentheses form a perfect square trinomial that is easily factored.

$$f(x) = 3\left[\left(x + \tfrac{2}{3}\right)^2 - \tfrac{28}{9}\right]$$

Finally, redistribute the 3.

$$f(x) = 3\left(x + \tfrac{2}{3}\right)^2 - \tfrac{28}{3}$$

This is a parabola that opens upward. It is also stretched by a factor of 3, so it will be narrower than all of our previous examples. The parabola is also shifted 2/3 units to the left, then 28/3 units downward, placing the vertex at $(-2/3, -28/3)$, as shown in **Figure 5**(a). The table in **Figure 5**(b) calculates two points to the right of the axis of symmetry, and mirror points on the left of the axis of symmetry make for an accurate plot of the parabola.



(a)

x	$y = 3(x+2/3)^2 - 28/3$
0	−8
1	−1

(b)

Figure 5. Plotting the graph of the quadratic function $f(x) = 3(x + 2/3)^2 - 28/3$.

2.2 Exercises

In **Exercises 1-8**, expand the binomial.

1. $\left(x+\frac{4}{5}\right)^2$

2. $\left(x-\frac{4}{5}\right)^2$

3. $(x+3)^2$

4. $(x+5)^2$

5. $(x-7)^2$

6. $\left(x-\frac{2}{5}\right)^2$

7. $(x-6)^2$

8. $\left(x-\frac{5}{2}\right)^2$

In **Exercises 9-16**, factor the perfect square trinomial.

9. $x^2-\frac{6}{5}x+\frac{9}{25}$

10. $x^2+5x+\frac{25}{4}$

11. $x^2-12x+36$

12. $x^2+3x+\frac{9}{4}$

13. $x^2+12x+36$

14. $x^2-\frac{3}{2}x+\frac{9}{16}$

15. $x^2+18x+81$

16. $x^2+10x+25$

In **Exercises 17-24**, transform the given quadratic function into vertex form $f(x)=(x-h)^2+k$ by completing the square.

17. $f(x)=x^2-x+8$

18. $f(x)=x^2+x-7$

19. $f(x)=x^2-5x-4$

20. $f(x)=x^2+7x-1$

21. $f(x)=x^2+2x-6$

22. $f(x)=x^2+4x+8$

23. $f(x)=x^2-9x+3$

24. $f(x)=x^2-7x+8$

In **Exercises 25-32**, transform the given quadratic function into vertex form $f(x)=a(x-h)^2+k$ by completing the square.

25. $f(x)=-2x^2-9x-3$

26. $f(x)=-4x^2-6x+1$

27. $f(x)=5x^2+5x+5$

28. $f(x)=3x^2-4x-6$

29. $f(x)=5x^2+7x-3$

30. $f(x)=5x^2+6x+4$

31. $f(x)=-x^2-x+4$

32. $f(x)=-3x^2-6x+4$

In **Exercises 33-38**, find the vertex of the graph of the given quadratic function.

33. $f(x) = -2x^2 + 5x + 3$

34. $f(x) = x^2 + 5x + 8$

35. $f(x) = -4x^2 - 4x + 1$

36. $f(x) = 5x^2 + 7x + 8$

37. $f(x) = 4x^2 + 2x + 8$

38. $f(x) = x^2 + x - 7$

In **Exercises 39-44**, find the axis of symmetry of the graph of the given quadratic function.

39. $f(x) = -5x^2 - 7x - 8$

40. $f(x) = x^2 + 6x + 3$

41. $f(x) = -2x^2 - 5x - 8$

42. $f(x) = -x^2 - 6x + 2$

43. $f(x) = -5x^2 + x + 6$

44. $f(x) = x^2 - 9x - 6$

For each of the quadratic functions in **Exercises 45-66**, perform each of the following tasks.

i. Use the technique of completing the square to place the given quadratic function in vertex form.
ii. Set up a coordinate system on a sheet of graph paper. Label and scale each axis.
iii. Draw the axis of symmetry and label it with its equation. Plot the vertex and label it with its coordinates.
iv. Set up a table near your coordinate system that calculates the coordinates of two points on either side of the axis of symmetry. Plot these points and their mirror images across the axis of symmetry. Draw the parabola and label it with its equation
v. Use the graph of the parabola to determine the domain and range of the quadratic function. Describe the domain and range using interval notation.

45. $f(x) = x^2 - 8x + 12$

46. $f(x) = x^2 + 4x - 1$

47. $f(x) = x^2 + 6x + 3$

48. $f(x) = x^2 - 4x + 1$

49. $f(x) = x^2 - 2x - 6$

50. $f(x) = x^2 + 10x + 23$

51. $f(x) = -x^2 + 6x - 4$

52. $f(x) = -x^2 - 6x - 3$

53. $f(x) = -x^2 - 10x - 21$

54. $f(x) = -x^2 + 12x - 33$

55. $f(x) = 2x^2 - 8x + 3$

56. $f(x) = 2x^2 + 8x + 4$

57. $f(x) = -2x^2 - 12x - 13$

58. $f(x) = -2x^2 + 24x - 70$

59. $f(x) = (1/2)x^2 - 4x + 5$

60. $f(x) = (1/2)x^2 + 4x + 6$

61. $f(x) = (-1/2)x^2 - 3x + 1/2$

62. $f(x) = (-1/2)x^2 + 4x - 2$

63. $f(x) = 2x^2 + 7x - 2$

64. $f(x) = -2x^2 - 5x - 4$

65. $f(x) = -3x^2 + 8x - 3$

66. $f(x) = 3x^2 + 4x - 6$

In **Exercises 67-72**, find the range of the given quadratic function. Express your answer in both interval and set notation.

67. $f(x) = -2x^2 + 4x + 3$

68. $f(x) = x^2 + 4x + 8$

69. $f(x) = 5x^2 + 4x + 4$

70. $f(x) = 3x^2 - 8x + 3$

71. $f(x) = -x^2 - 2x - 7$

72. $f(x) = x^2 + x + 9$

Drill for Skill. In **Exercises 73-76**, evaluate the function at the given value b.

73. $f(x) = 9x^2 - 9x + 4$; $b = -6$

74. $f(x) = -12x^2 + 5x + 2$; $b = -3$

75. $f(x) = 4x^2 - 6x - 4$; $b = 11$

76. $f(x) = -2x^2 - 11x - 10$; $b = -12$

Drill for Skill. In **Exercises 77-80**, evaluate the function at the given expression.

77. Evaluate $f(x+4)$ if $f(x) = -5x^2 + 4x + 2$.

78. Evaluate $f(-4x-5)$ if $f(x) = 4x^2 + x + 1$.

79. Evaluate $f(4x-1)$ if $f(x) = 4x^2 + 3x - 3$.

80. Evaluate $f(-5x-3)$ if $f(x) = -4x^2 + x + 4$.

2.2 Answers

1. $x^2 + \frac{8}{5}x + \frac{16}{25}$

3. $x^2 + 6x + 9$

5. $x^2 - 14x + 49$

7. $x^2 - 12x + 36$

9. $\left(x - \frac{3}{5}\right)^2$

11. $(x - 6)^2$

13. $(x + 6)^2$

15. $(x + 9)^2$

17. $\left(x - \frac{1}{2}\right)^2 + \frac{31}{4}$

19. $\left(x - \frac{5}{2}\right)^2 - \frac{41}{4}$

21. $(x + 1)^2 - 7$

23. $\left(x - \frac{9}{2}\right)^2 - \frac{69}{4}$

25. $-2\left(x + \frac{9}{4}\right)^2 + \frac{57}{8}$

27. $5\left(x + \frac{1}{2}\right)^2 + \frac{15}{4}$

29. $5\left(x + \frac{7}{10}\right)^2 - \frac{109}{20}$

31. $-1\left(x + \frac{1}{2}\right)^2 + \frac{17}{4}$

33. $\left(\frac{5}{4}, \frac{49}{8}\right)$

35. $\left(-\frac{1}{2}, 2\right)$

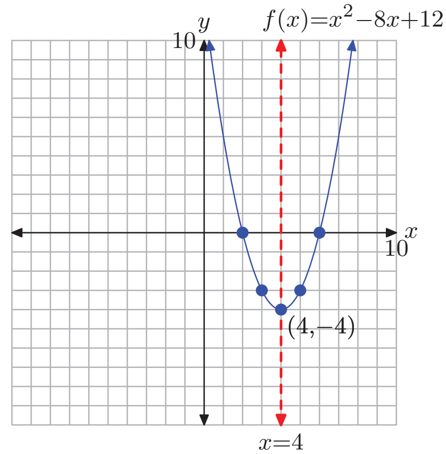
37. $\left(-\frac{1}{4}, \frac{31}{4}\right)$

39. $x = -\frac{7}{10}$

41. $x = -\frac{5}{4}$

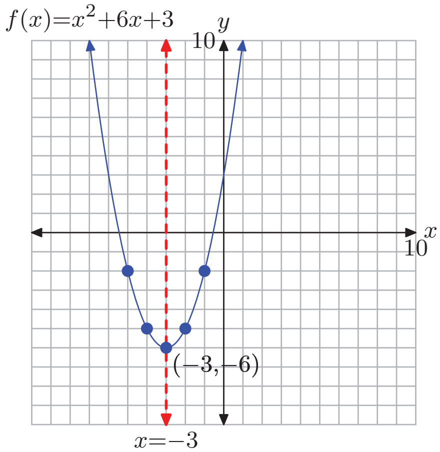
43. $x = \frac{1}{10}$

45. $f(x) = (x - 4)^2 - 4$



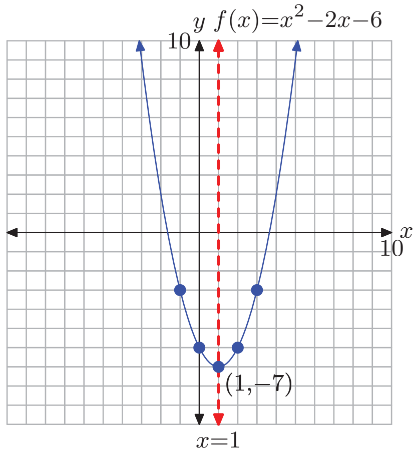
Domain $= \mathbb{R}$, Range $= [-4, \infty)$

47. $f(x) = (x+3)^2 - 6$



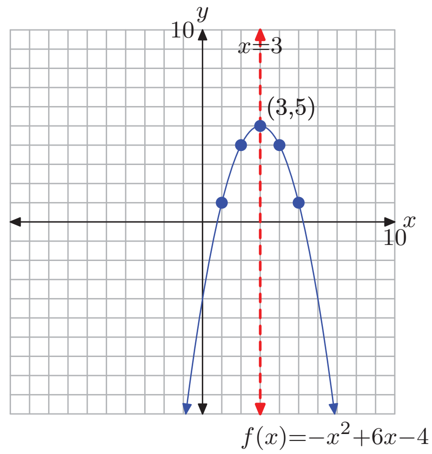
Domain $= \mathbb{R}$, Range $= [-6, \infty)$

49. $f(x) = (x-1)^2 - 7$



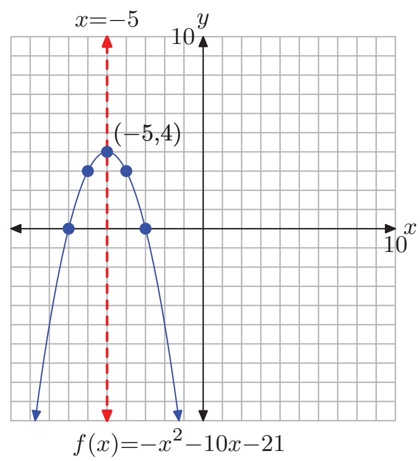
Domain $= \mathbb{R}$, Range $= [-7, \infty)$

51. $f(x) = -(x-3)^2 + 5$



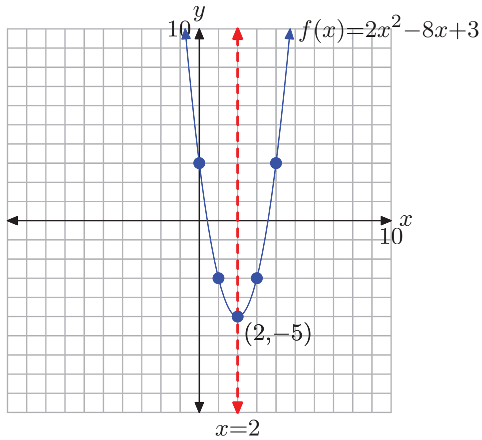
Domain $= \mathbb{R}$, Range $= (-\infty, 5]$

53. $f(x) = -(x+5)^2 + 4$



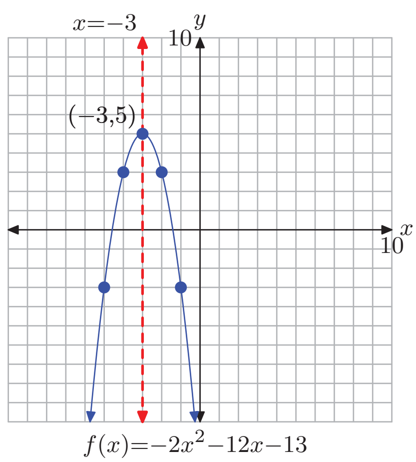
Domain $= \mathbb{R}$, Range $= (-\infty, 4]$

55. $f(x) = 2(x-2)^2 - 5$



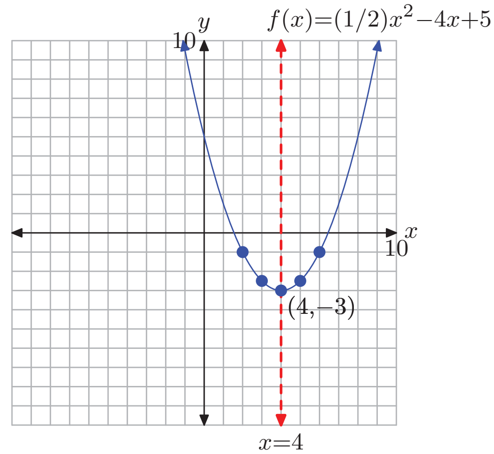
Domain $= \mathbb{R}$, Range $= [-5, \infty)$

57. $f(x) = -2(x+3)^2 + 5$



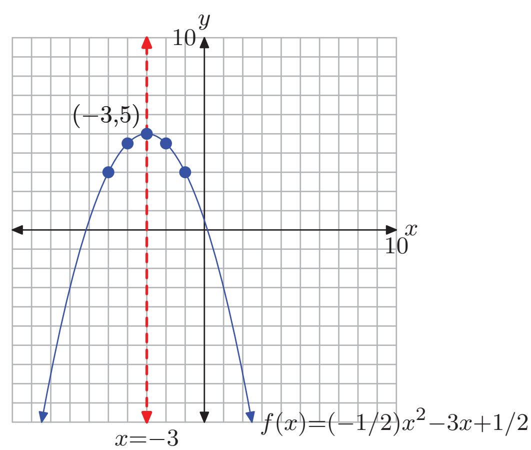
Domain $= \mathbb{R}$, Range $= (-\infty, 5]$

59. $f(x) = (1/2)(x-4)^2 - 3$



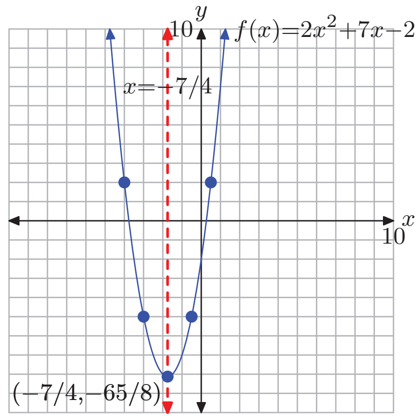
Domain $= \mathbb{R}$, Range $= [-3, \infty)$

61. $f(x) = (-1/2)(x+3)^2 + 5$



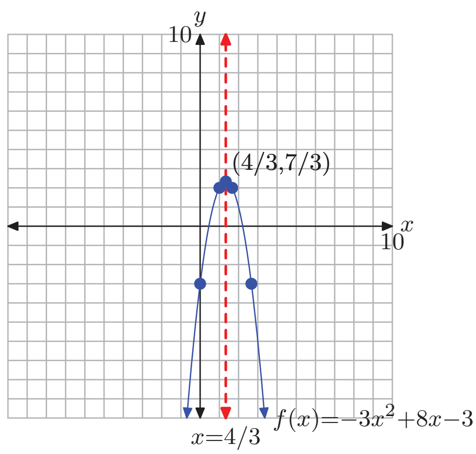
Domain $= \mathbb{R}$, Range $= (-\infty, 5])$

63. $f(x) = 2(x + 7/4)^2 - 65/8$



Domain $= \mathbb{R}$, Range $= [-65/8, \infty)$

65. $f(x) = -3(x - 4/3)^2 + 7/3$



Domain $= \mathbb{R}$, Range $= (-\infty, 7/3]$

67. $(-\infty, 5] = \{x \,|\, x \leq 5\}$

69. $\left[\dfrac{16}{5}, \infty\right) = \left\{x \,\middle|\, x \geq \dfrac{16}{5}\right\}$

71. $(-\infty, -6] = \{x \,|\, x \leq -6\}$

73. 382

75. 414

77. $-5x^2 - 36x - 62$

79. $64x^2 - 20x - 2$

2.3 Zeros of the Quadratic

We've seen how vertex form and intelligent use of the axis of symmetry can help to draw an accurate graph of the quadratic function defined by the equation $f(x) = ax^2+bx+c$. When drawing the graph of the parabola it is helpful to know where the graph of the parabola crosses the x-axis. That is the primary goal of this section, to find the zero crossings or x-intercepts of the parabola.

Before we begin, you'll need to review the techniques that will enable you to factor the quadratic expression $ax^2 + bx + c$.

Factoring $ax^2 + bx + c$ when $a = 1$

Our intent in this section is to provide a quick review of techniques used to factor quadratic trinomials. We begin by showing how to factor trinomials having the form $ax^2 + bx + c$, where the leading coefficient is $a = 1$; that is, trinomials having the form x^2+bx+c. In the next section, we will address the technique used to factor ax^2+bx+c when $a \neq 1$.

Let's begin with an example.

I **Example 1.** *Factor $x^2 + 16x - 36$.*

Note that the leading coefficient, the coefficient of x^2, is a 1. This is an important observation, because the technique presented here will not work when the leading coefficient does not equal 1.

Note the constant term of the trinomial $x^2 + 16x - 36$ is -36. List all integer pairs whose product equals -36.

1, −36		−1, 36
2, −18		**[−2, 18]**
3, −12		−3, 12
4, −9		−4, 9
6, −6		−6, 6

Note that we've framed the pair -2, 18. We've done this because the sum of this pair of integers equals the coefficient of x in the trinomial expression $x^2 + 16x - 36$. Use this framed pair to factor the trinomial.

$$x^2 + 16x - 36 = (x - 2)(x + 18)$$

It is important that you check your result. Use the distributive property to multiply.

$$\begin{aligned}(x - 2)(x + 18) &= x(x + 18) - 2(x + 18) \\ &= x^2 + 18x - 2x - 36 \\ &= x^2 + 16x - 36\end{aligned}$$

[6] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

Thus, our factorization is correct.

Let's summarize the technique.

Algorithm. To factor the quadratic $x^2 + bx + c$, proceed as follows:

1. List all the integer pairs whose product equals c.
2. Circle or frame the pair whose sum equals the coefficient of x, namely b. Use this pair to factor the trinomial.

Let's look at another example.

I **Example 2.** *Factor the trinomial* $x^2 - 25x - 84$.

List all the integer pairs whose product is -84.

1, −84		−1, 84
2, −42		−2, 42
[3, −28]		−3, 28
4, −21		−4, 21
6, −14		−6, 14
7, −12		−7, 12

We've framed the pair whose sum equals the coefficient of x, namely -25. Use this pair to factor the trinomial.

$$x^2 - 25x - 84 = (x + 3)(x - 28)$$

Check.

$$\begin{aligned}(x + 3)(x - 28) &= x(x - 28) + 3(x - 28) \\ &= x^2 - 28x + 3x - 84 \\ &= x^2 - 25x - 84\end{aligned}$$

With experience, there are a number of ideas that will quicken the process.

- As you are listing the integer pairs, should you happen to note that the current pair has the appropriate sum, there is no need to list the remaining integer pairs. Simply halt the process of listing the integer pairs and use the current pair to factor the trinomial.
- Some students are perfectly happy being asked "Can you think of an integer pair whose product is c and whose sum is b (where b and c refer to the coefficients of $x^2 + bx + c$)?" If you can pick the pair "out of the air" like this, all is well and good.

Use the integer pair to factor the trinomial and don't bother listing any integer pairs.

Now, let's investigate how to proceed when the leading coefficient is not 1.

Factoring $ax^2 + bx + c$ when $a \neq 1$

When $a \neq 1$, we use a technique called the *ac-test* to factor the trinomial $ax^2 + bx + c$. The process is best explained with an example.

I **Example 3.** *Factor* $2x^2 + 13x - 24$.

Note that the leading coefficient does not equal 1. Indeed, the coefficient of x^2 in this example is a 2. Therefore, the technique of the previous examples will not work. Thus, we turn to a similar technique called the *ac-test*.

First, compare

$$2x^2 + 13x - 24 \qquad \text{and} \qquad ax^2 + bx + c$$

and note that $a = 2$, $b = 13$, and $c = -24$. Compute the product of a and c. This is how the technique earns its name "*ac*-test."

$$ac = (2)(-24) = -48$$

List all integer pairs whose product is $ac = -48$.

1, −48		−1, 48
2, −24		−2, 24
3, −16		**[−3, 16]**
4, −12		−4, 12
6, −8		−6, 8

We've framed the pair whose sum is $b = 13$. The next step is to rewrite the trinomial $2x^2 + 13x - 24$, splitting the middle term into a sum, using our framed integer pair.

$$2x^2 + 13x - 24 = 2x^2 - 3x + 16x - 24$$

We factor an x out of the first two terms, then an 8 out of the last two terms. This process is called *factoring by grouping*.

$$2x^2 - 3x + 16x - 24 = x(2x - 3) + 8(2x - 3)$$

We now factor out a common factor of $2x - 3$.

$$x(2x - 3) + 8(2x - 3) = (x + 8)(2x - 3)$$

It's helpful to see the complete process as a coherent unit.

$$\begin{aligned}2x^2 + 13x - 24 &= 2x^2 - 3x + 16x - 24\\ &= x(2x-3) + 8(2x-3)\\ &= (x+8)(2x-3)\end{aligned}$$

Check. Again, it is important to check the answer by multiplication.

$$\begin{aligned}(x+8)(2x-3) &= x(2x-3) + 8(2x-3)\\ &= 2x^2 - 3x + 16x - 24\\ &= 2x^2 + 13x - 24\end{aligned}$$

Because this is the original trinomial, our solution checks.[7]

Let's summarize this process.

Algorithm: ac-Test. To factor the quadratic $ax^2 + bx + c$, proceed as follows:

1. List all integer pairs whose product equals ac.
2. Circle or frame the pair whose sum equals the coefficient of x, namely b.
3. Use the circled pair to express the middle term bx as a sum.
4. Factor by "grouping."

Let's look at another example.

I **Example 4.** *Factor* $3x^2 + 34x - 24$.

Compare

$$3x^2 + 34x - 24 \quad \text{and} \quad ax^2 + bx + c$$

and note that $a = 3$, $b = 34$ and $c = -24$. List all integer pairs whose product equals $ac = (3)(-24) = -72$.

1, −72		−1, 72
2, −36		$\boxed{-2, 36}$
3, −24		−3, 24
4, −18		−4, 18
6, −12		−6, 12
8, −9		−8, 9

We've framed the pair whose sum is the same as $b = 34$, the coefficient of x in $3x^2 + 34x - 24$. Again, possible shortcuts are possible. If you can "think" of a pair whose product is $ac = -72$ and whose sum is $b = 34$, then it is not necessary to list any integer pairs. Alternatively, if you come across the needed pair as you are listing

[7] If you check a number of your results, it will soon become apparent why the *ac*-test works so well.

them, then you can halt the process. There is no need to list the remaining pairs if you have the one you need.

Use the framed pair to express the middle term as a sum, then factor by grouping.

$$\begin{aligned} 3x^2 + 34x - 24 &= 3x^2 - 2x + 36x - 24 \\ &= x(3x-2) + 12(3x-2) \\ &= (x+12)(3x-2) \end{aligned}$$

We leave it to the reader to check this result.

Intercepts

The points where the graph of a function crosses the x-axis are called the *x-intercepts* of graph of the function. Consider the graph of the quadratic function f in **Figure 1**.

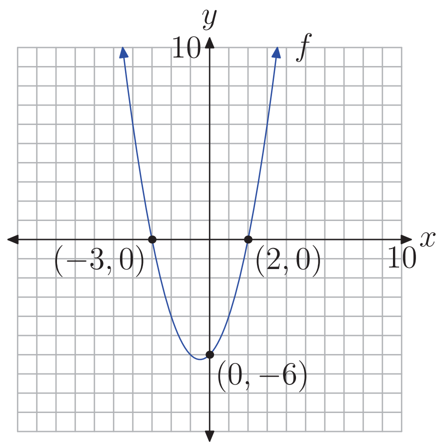


Figure 1. The x- and y-intercepts are key features of any graph.

Note that the graph of the f crosses the x-axis at $(-3,0)$ and $(2,0)$. These are the x-intercepts of the parabola. Note that the y-coordinate of each x-intercept is zero.

In function notation, the solutions of $f(x) = 0$ (note the similarity to $y = 0$) are the x-coordinates of the points where the graph of f crosses the x-axis. Analyzing the graph of f in **Figure 1**, we see that both -3 and 2 are solutions of $f(x) = 0$.

Thus, the process for finding the x-intercepts is clear.

Finding x-intercepts. To find the x-intercepts of the graph of any function, set $y = 0$ and solve for x. Alternatively, if function notation is used, set $f(x) = 0$ and solve for x.

Let's look at an example.

I **Example 5.** *Find the x-intercepts of the graph of the quadratic function defined by $y = x^2 + 2x - 48$.*

To find the x-intercepts, first set $y = 0$.

$$0 = x^2 + 2x - 48$$

Next, factor the trinomial on the right. Note that the coefficient of x^2 is 1. We need only think of two integers whose product equals the constant term -48 and whose sum equals the coefficient of x, namely 2. The numbers 8 and -6 come to mind, so the trinomial factors as follows (readers should check this result).

$$0 = (x + 8)(x - 6)$$

To complete the solution, we need to use an important property of the real numbers called the *zero product property*.

Zero Product Property. If a and b are any real numbers such that

$$ab = 0,$$

then either $a = 0$ or $b = 0$.

In our case, we have $0 = (x + 8)(x - 6)$. Therefore, it must be the case that either

$$x + 8 = 0 \qquad \text{or} \qquad x - 6 = 0.$$

These equations can be solved independently to produce

$$x = -8 \qquad \text{or} \qquad x = 6.$$

Thus, the x-intercepts of the graph of $y = x^2 + 2x - 48$ are located at $(-8, 0)$ and $(6, 0)$.

Let's look at another example.

I **Example 6.** *Find the x-intercepts of the graph of the quadratic function $f(x) = 2x^2 - 7x - 15$.*

To find the x-intercepts of the graph of the quadratic function f, we begin by setting

$$f(x) = 0.$$

Of course, $f(x) = 2x^2 - 7x - 15$, so we can substitute to obtain

$$2x^2 - 7x - 15 = 0.$$

We will now use the ac-test to factor the trinomial on the left. Note that $ac = (2)(-15) = -30$. List the integer pairs whose products equal -30.

1, −30		−1, 20
2, −15		−2, 15
3, −10		−3, 10
5, −6		−5, 6

Note that the framed pair sum to the coefficient of x in $2x^2 - 7x - 15$. Use the framed pair to express the middle term as a sum, then factor by grouping.

$$\begin{aligned} 2x^2 - 7x - 15 &= 0 \\ 2x^2 + 3x - 10x - 15 &= 0 \\ x(2x+3) - 5(2x+3) &= 0 \\ (x-5)(2x+3) &= 0 \end{aligned}$$

Now we can use the zero product property. Either

$$x - 5 = 0 \qquad \text{or} \qquad 2x + 3 = 0.$$

Each of these can be solved independently to obtain

$$x = 5 \qquad \text{or} \qquad x = -3/2.$$

Thus, the x-intercepts of the graph of the quadratic function $f(x) = 2x^2 - 7x - 15$ are located at $(-3/2, 0)$ and $(5, 0)$.

One more definition is in order.

Definition 7. Zeros of a Function. *The solutions of $f(x) = 0$ are called the* **zeros** *of the function f.*

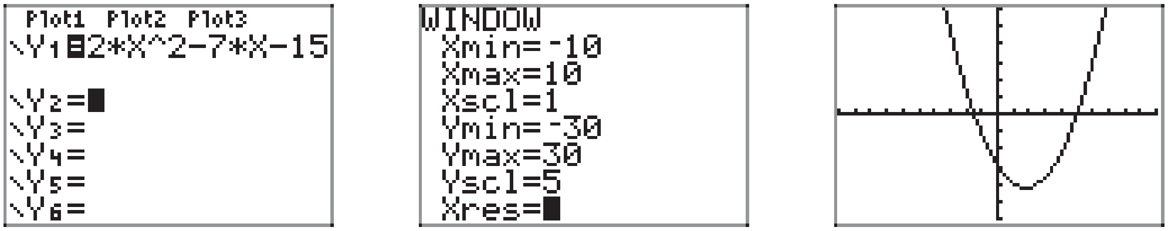
Thus, in the last example, both $-3/2$ and 5 are zeros of the quadratic function $f(x) = 2x^2 - 7x - 15$. Note the intimate relationship between the zeros of the quadratic function and the x-intercepts of the graph. Note that $-3/2$ is a zero and $(-3/2, 0)$ is an x-intercept. Similarly, 5 is a zero and $(5, 0)$ is an x-intercept.

The graphing calculator can be used to find the zeros of a function.

I **Example 8.** *Use the graphing calculator to find the zeros of the function $f(x) = 2x^2 - 7x - 15$.*

Enter the function $f(x) = 2x^2 - 7x - 15$ into `Y1` in the `Y=` menu; then adjust the window parameters as shown in **Figure 2**(b). Push the `GRAPH` button to produce the parabola shown in **Figure 2**(c).

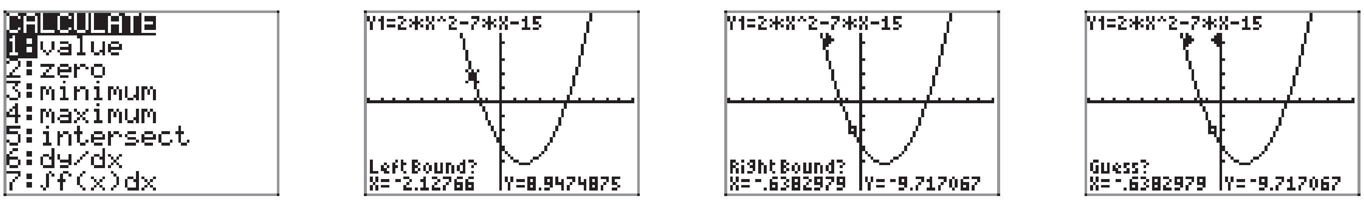
To find a zero of the function, proceed as follows:



(a) (b) (c)

Figure 2. Plotting the quadratic function $f(x) = 2x^2 - 7x - 15$.

- Press `2nd TRACE` to open the `CALCULATE` window shown in **Figure 3**(a). From this menu, select `2:zero`.
- The calculator responds by asking for a "Left bound." Use the arrow keys to move the cursor slightly to the left of the leftmost x-intercept, as shown in **Figure 3**(b). Press the `ENTER` key.
- The calculator responds by asking for a "Right bound." Use the arrow keys to move the cursor slightly to the right of the leftmost x-intercept, as shown in **Figure 3**(c). Press the `ENTER` key.
- The calculator responds by asking for a "Guess." You may use the arrow keys to select a starting x-value any where between the left- and right-bounds you selected (note that the calculator marks these on the screen in **Figure 3**(d)). However, the cursor already lies between these marks, so we typically just hit `ENTER` at this point. We suggest you do so also.



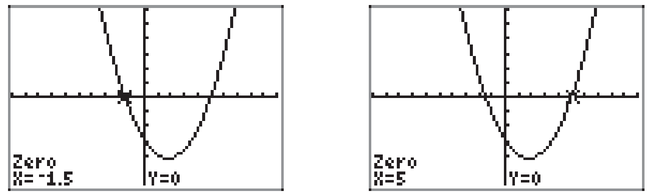
(a) (b) (c) (d)

Figure 3. Using the `zero` utility to find an x-intercept.

The calculator responds by marking the x-intercept and reporting its x-value at the bottom of the screen, as shown in **Figure 4**(a). This is one of the zeros of the function. Note that this value of -1.5 agrees nicely with our hand calculated result $-3/2$ in **Example 6**. We followed precisely the same procedure outlined above to find the second x-intercept shown in **Figure 4**(b). Note that it also agrees with the hand calculated solution of **Example 6**.

In a similar vein, the point where the graph of a function crosses the y-axis is called the *y-intercept* of the graph of the function. In **Figure 1** the y-intercept of the parabola is $(0, -6)$. Note that the x-coordinate of this y-intercept is zero.

Thus, the process for finding y-intercepts should be clear.



(a) (b)

Figure 4. The zeros of $f(x) - 2x^2 - 7x - 15$.

> **Finding y-intercepts.** To find the y-intercepts of the graph of any function, set $x = 0$ and solve for y. Alternatively, if function notation is used, simply evaluate $f(0)$.

I **Example 9.** *Find the y-intercept of the quadratic function defined by* $f(x) = x^2 - 3x - 11$.

Evaluate the function at $x = 0$.

$$f(0) = (0)^2 - 3(0) - 11 = -11.$$

The coordinates of the y-intercept are $(0, -11)$.

Putting it All Together

We will find both x- and y-intercepts extremely useful when drawing the graph of a quadratic function.

I **Example 10.** *Place the quadratic function* $y = x^2 + 2x - 24$ *in vertex form. Plot the vertex and axis of symmetry and label them with their coordinates and equation, respectively. Find and plot the x- and y-intercepts of the parabola and label them with their coordinates.*

Take half of the coefficient of x, square, then add and subtract this amount to balance the equation. Factor and combine coefficients.

$$y = x^2 + 2x + 1 - 1 - 24$$
$$y = (x + 1)^2 - 25$$

The graph is a parabola that opens upward; it is shifted 1 unit to the left and 25 units downward. This information is enough to plot and label the vertex, then plot and label the axis of symmetry, as shown in **Figure 5**(a).

To find the x-intercepts, let $y = 0$ in $y = x^2 + 2x - 24$.

$$0 = x^2 + 2x - 24$$

The leading coefficient is a 1. The integer pair −4 and 6 has product −24 and sum 2. Thus, the right-hand side factors as follows.

$$0 = (x + 6)(x - 4)$$

In order that this product equals zero, either

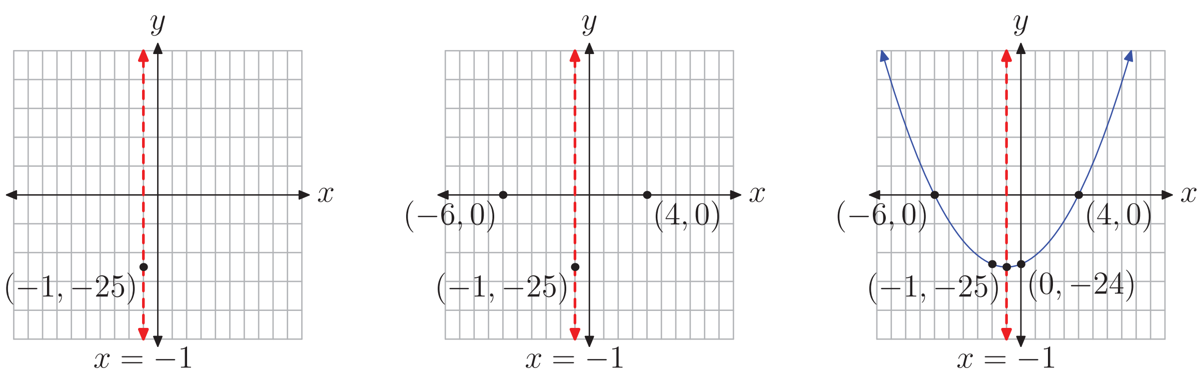
$$x + 6 = 0 \qquad \text{or} \qquad x - 4 = 0.$$

Solve each of these linear equations independently.

$$x = -6 \qquad \text{or} \qquad x = 4.$$

Recall that we let $y = 0$. We've found two solutions, $x = -6$ and $x = 4$. Thus, we have x-intercepts at $(-6, 0)$ and $(4, 0)$, as pictured in **Figure 5**(b).

Finally, to find the y-intercept, let $x = 0$ in $y = x^2 + 2x - 24$. With this substitution, $y = -24$. Thus, the y-intercept is $(0, -24)$, as pictured in **Figure 5**(c). Note that we've also included the mirror image of the y-intercept across the axis of symmetry.



(a) Plotting the vertex and axis of symmetry.

(b) Adding the x-intercepts provides added accuracy.

(c) Adding the y-intercept and its mirror image provides an excellent final graph.

Figure 5.

Let's look at one final example.

I **Example 11.** *Plot the parabola represented by the equation* $f(x) = -2x^2 - 7x + 15$. *Plot and label the vertex, axis of symmetry, and the* x*- and* y*-intercepts.*

First, factor out a −2.

$$f(x) = -2\left[x^2 + \frac{7}{2}x - \frac{15}{2}\right]$$

Half[8] of $7/2$ is $7/4$. Squared, this amounts to $49/16$. Add and subtract this last amount to keep the equation balanced.

$$f(x) = -2\left[x^2 + \frac{7}{2}x + \frac{49}{16} - \frac{49}{16} - \frac{15}{2}\right]$$

The first three terms inside the parentheses form a perfect square trinomial. The last two constants are combined with a common denominator.

$$f(x) = -2\left[\left(x^2 + \frac{7}{2}x + \frac{49}{16}\right) - \frac{49}{16} - \frac{120}{16}\right]$$
$$f(x) = -2\left[\left(x + \frac{7}{4}\right)^2 - \frac{169}{16}\right]$$

Finally, redistribute the -2.

$$f(x) = -2\left(x + \frac{7}{4}\right)^2 + \frac{169}{8}$$

The graph of this last equation is a parabola that opens downward, translated $7/4$ units to the left and $169/8$ units upward. This is enough information to plot and label the vertex and axis of symmetry, as shown in **Figure 6**(a).

To find the y-intercepts, set $f(x) = 0$ in $f(x) = -2x^2 - 7x + 15$. We will also multiply both sides of the resulting equation by -1.

$$0 = -2x^2 - 7x + 15$$
$$0 = 2x^2 + 7x - 15$$

After comparing $2x^2 + 7x - 15$ with $ax^2 + bx + c$, we note that the integer pair -3 and 10 have product equal to $ac = -30$ and sum equal to $b = 7$. Use this pair to express the middle term of $2x^2 + 7x - 15$ as a sum and then factor by grouping.

$$0 = 2x^2 - 3x + 10x - 15$$
$$0 = x(2x - 3) + 5(2x - 3)$$
$$0 = (x + 5)(2x - 3)$$

By the *zero product property*, either

$$x + 5 = 0 \qquad \text{or} \qquad 2x - 3 = 0.$$

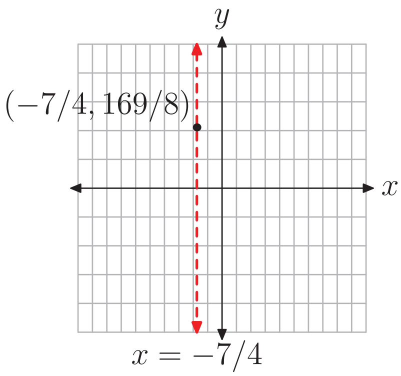
Solve these linear equations independently.

$$x = -5 \qquad \text{or} \qquad x = \frac{3}{2}$$

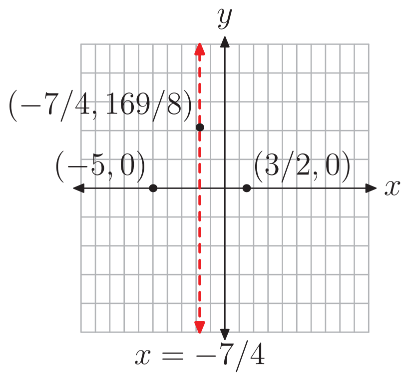
These x-values are the zeros of f (they make $f(x) = 0$), so we have x-intercepts at $(-5, 0)$ and $(3/2, 0)$, as shown in **Figure 6**(b).

---

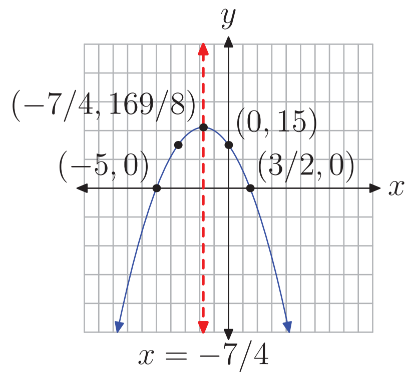
8 $\frac{1}{2} \cdot \frac{7}{2} = \frac{7}{4}$



(a) Plotting the vertex and axis of symmetry.



(b) Adding the x-intercepts provides added accuracy.



(c) Adding the y-intercept and its mirror image provide an excellent final graph.

Figure 6.

Finally, to find the y-intercept, set $x = 0$ in $f(x) = -2x^2 - 7x + 15$ to get $f(0) = 15$. Note the positioning of the y-intercept $(0, 15)$ and its mirror image across the axis of symmetry in **Figure 6**(c).

2.3 *Exercises*

In **Exercises 1-8**, factor the given quadratic polynomial.

1. $x^2 + 9x + 14$

2. $x^2 + 6x + 5$

3. $x^2 + 10x + 9$

4. $x^2 + 4x - 21$

5. $x^2 - 4x - 5$

6. $x^2 + 7x - 8$

7. $x^2 - 7x + 12$

8. $x^2 + 5x - 24$

In **Exercises 9-16**, find the zeros of the given quadratic function.

9. $f(x) = x^2 - 2x - 15$

10. $f(x) = x^2 + 4x - 32$

11. $f(x) = x^2 + 10x - 39$

12. $f(x) = x^2 + 4x - 45$

13. $f(x) = x^2 - 14x + 40$

14. $f(x) = x^2 - 5x - 14$

15. $f(x) = x^2 + 9x - 36$

16. $f(x) = x^2 + 11x - 26$

In **Exercises 17-22**, perform each of the following tasks for the quadratic functions.

i. Load the function into `Y1` of the `Y=` of your graphing calculator. Adjust the window parameters so that the vertex is visible in the viewing window.
ii. Set up a coordinate system on your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Make a reasonable copy of the image in the viewing window of your calculator on this coordinate system and label it with its equation.
iii. Use the `zero` utility on your graphing calculator to find the zeros of the function. Use these results to plot the x-intercepts on your coordinate system and label them with their coordinates.
iv. Use a strictly algebraic technique (no calculator) to find the zeros of the given quadratic function. Show your work next to your coordinate system. Be stubborn! Work the problem until your algebraic and graphically zeros are a reasonable match.

17. $f(x) = x^2 + 5x - 14$

18. $f(x) = x^2 + x - 20$

19. $f(x) = -x^2 + 3x + 18$

20. $f(x) = -x^2 + 3x + 40$

21. $f(x) = x^2 - 16x - 36$

22. $f(x) = x^2 + 4x - 96$

In **Exercises 23-30**, perform each of the following tasks for the given quadratic function.

i. Set up a coordinate system on graph paper. Label and scale each axis. *Remember to draw all lines with a ruler.*
ii. Use the technique of completing the square to place the quadratic function in vertex form. Plot the vertex on your coordinate system and label it with its coordinates. Draw the axis of symmetry on your coordinate system and label it with its equation.
iii. Use a strictly algebraic technique (no calculators) to find the x-intercepts of the graph of the given quadratic function. Plot them on your coordinate system and label them with their coordinates.
iv. Find the y-intercept of the graph of the quadratic function. Plot the y-intercept on your coordinate system and its mirror image across the axis of symmetry, then label these points with their coordinates.
v. Using all the information plotted, draw the graph of the quadratic function and label it with the vertex form of its equation. Use interval notation to describe the domain and range of the quadratic function.

23. $f(x) = x^2 + 2x - 8$

24. $f(x) = x^2 - 6x + 8$

25. $f(x) = x^2 + 4x - 12$

26. $f(x) = x^2 + 8x + 12$

27. $f(x) = -x^2 - 2x + 8$

28. $f(x) = -x^2 - 2x + 24$

29. $f(x) = -x^2 - 8x + 48$

30. $f(x) = -x^2 - 8x + 20$

In **Exercises 31-38**, factor the given quadratic polynomial.

31. $42x^2 + 5x - 2$

32. $3x^2 + 7x - 20$

33. $5x^2 - 19x + 12$

34. $54x^2 - 3x - 1$

35. $-4x^2 + 9x - 5$

36. $3x^2 - 5x - 12$

37. $2x^2 - 3x - 35$

38. $-6x^2 + 25x + 9$

In **Exercises 39-46**, find the zeros of the given quadratic functions.

39. $f(x) = 2x^2 - 3x - 20$

40. $f(x) = 2x^2 - 7x - 30$

41. $f(x) = -2x^2 + x + 28$

42. $f(x) = -2x^2 + 15x - 22$

43. $f(x) = 3x^2 - 20x + 12$

44. $f(x) = 4x^2 + 11x - 20$

45. $f(x) = -4x^2 + 4x + 15$

46. $f(x) = -6x^2 - x + 12$

In **Exercises 47-52**, perform each of the following tasks for the given quadratic functions.

i. Load the function into `Y1` of the `Y=` of your graphing calculator. Adjust the window parameters so that the vertex is visible in the viewing window.
ii. Set up a coordinate system on your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Make a reasonable copy of the image in the viewing window of your calculator on this coordinate system and label it with its equation.
iii. Use the `zero` utility on your graphing calculator to find the zeros of the function. Use these results to plot the x-intercepts on your coordinate system and label them with their coordinates.
iv. Use a strictly algebraic technique (no calculator) to find the zeros of the given quadratic function. Show your work next to your coordinate system. Be stubborn! Work the problem until your algebraic and graphically zeros are a reasonable match.

47. $f(x) = 2x^2 + 3x - 35$

48. $f(x) = 2x^2 - 5x - 42$

49. $f(x) = -2x^2 + 5x + 33$

50. $f(x) = -2x^2 - 5x + 52$

51. $f(x) = 4x^2 - 24x - 13$

52. $f(x) = 4x^2 + 24x - 45$

In **Exercises 53-60**, perform each of the following tasks for the given quadratic functions.

i. Set up a coordinate system on graph paper. Label and scale each axis. *Remember to draw all lines with a ruler.*
ii. Use the technique of completing the square to place the quadratic function in vertex form. Plot the vertex on your coordinate system and label it with its coordinates. Draw the axis of symmetry on your coordinate system and label it with its equation.
iii. Use a strictly algebraic method (no calculators) to find the x-intercepts of the graph of the quadratic function. Plot them on your coordinate system and label them with their coordinates.
iv. Find the y-intercept of the graph of the quadratic function. Plot the y-intercept on your coordinate system and its mirror image across the axis of symmetry, then label these points with their coordinates.
v. Using all the information plotted, draw the graph of the quadratic function and label it with the vertex form of its equation. Use interval notation to describe the domain and range of the quadratic function.

53. $f(x) = 2x^2 - 8x - 24$

54. $f(x) = 2x^2 - 4x - 6$

55. $f(x) = -2x^2 - 4x + 16$

56. $f(x) = -2x^2 - 16x + 40$

57. $f(x) = 3x^2 + 18x - 48$

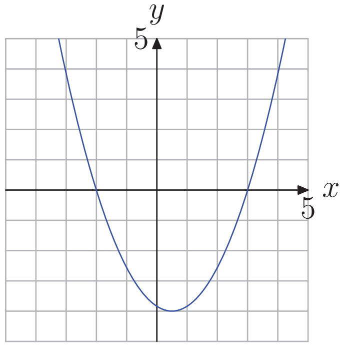
58. $f(x) = 3x^2 + 18x - 216$

59. $f(x) = 2x^2 + 10x - 48$

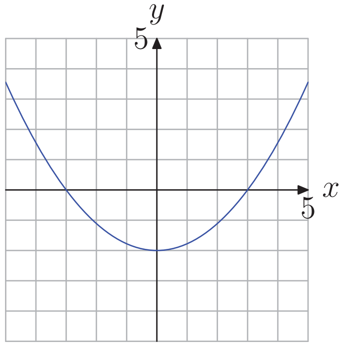
60. $f(x) = 2x^2 - 10x - 100$

In **Exercises 61-66**, Use the graph of $f(x) = ax^2 + bx + c$ shown to find all solutions of the equation $f(x) = 0$. (Note: Every solution is an integer.)

61.



62.



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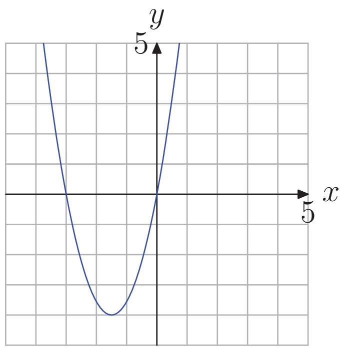

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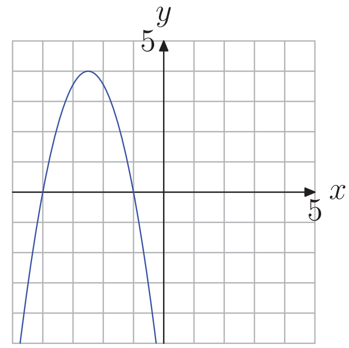

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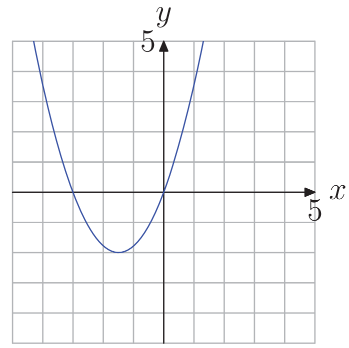
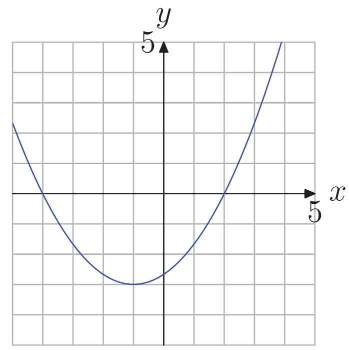


66.



2.3 Answers

1. $(x+2)(x+7)$

3. $(x+9)(x+1)$

5. $(x-5)(x+1)$

7. $(x-4)(x-3)$

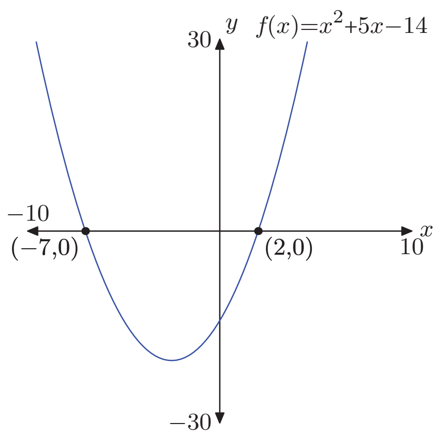
9. Zeros: $x=-3$, $x=5$

11. Zeros: $x=-13$, $x=3$

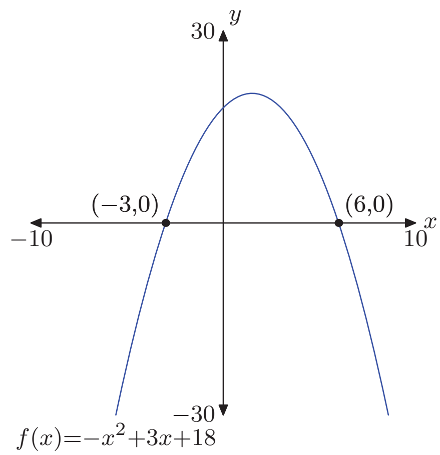
13. Zeros: $x=4$, $x=10$

15. Zeros: $x=-12$, $x=3$

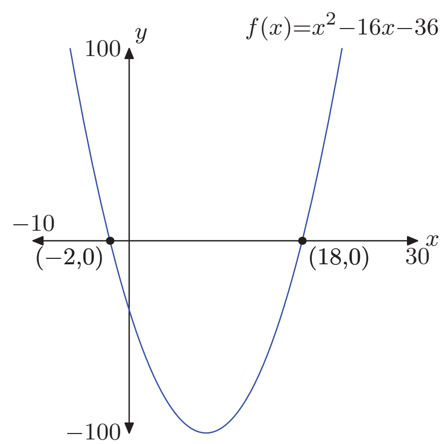
17.



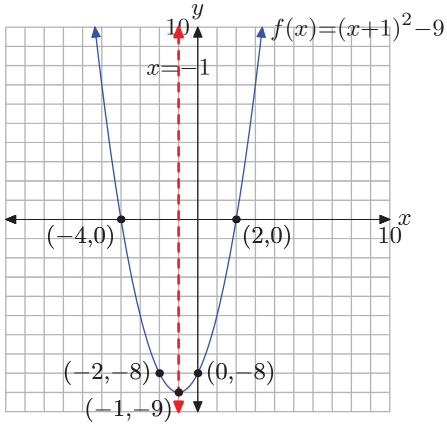
19.



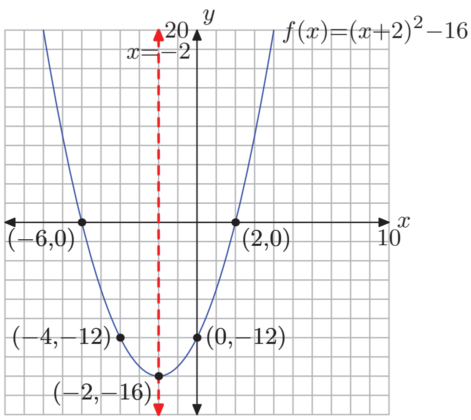
21.



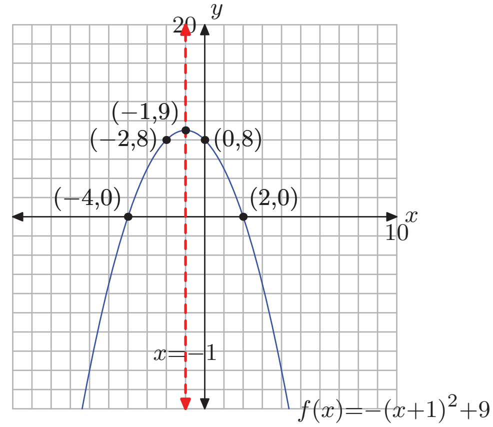
23. Domain = $(-\infty, \infty)$, Range = $[-9, \infty)$



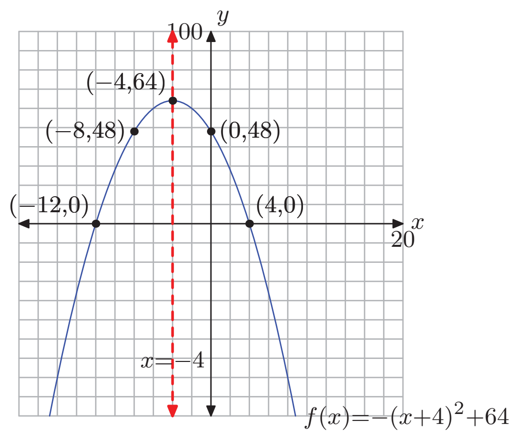
25. Domain = $(-\infty, \infty)$, Range = $[-16, \infty)$



27. Domain = $(-\infty, \infty)$, Range = $(-\infty, 9]$



29. Domain = $(-\infty, \infty)$, Range = $(-\infty, 64]$



31. $(7x + 2)(6x - 1)$

33. $(x - 3)(5x - 4)$

35. $(4x - 5)(-x + 1)$

37. $(2x + 7)(x - 5)$

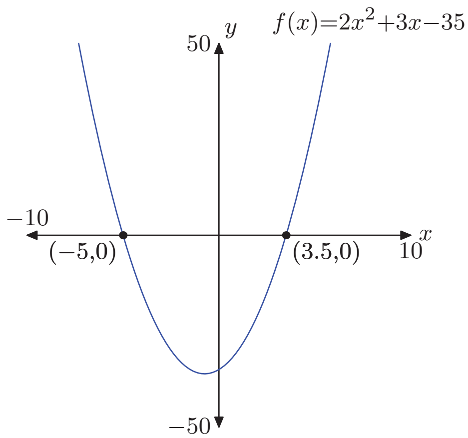
39. Zeros: $x = -5/2$, $x = 4$

41. Zeros: $x = -7/2$, $x = 4$

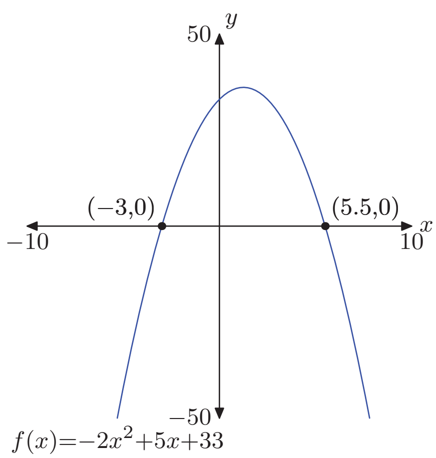
43. Zeros: $x = 2/3$, $x = 6$

45. Zeros: $x = -3/2$, $x = 5/2$

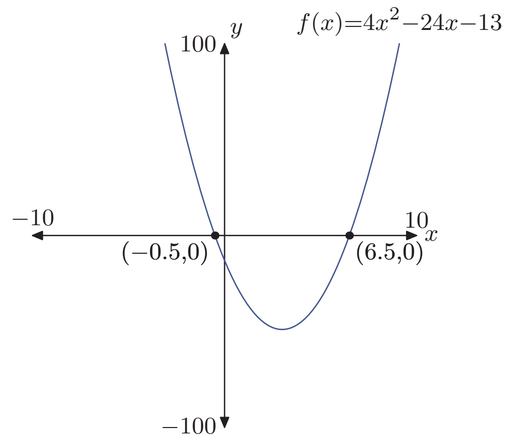
47.



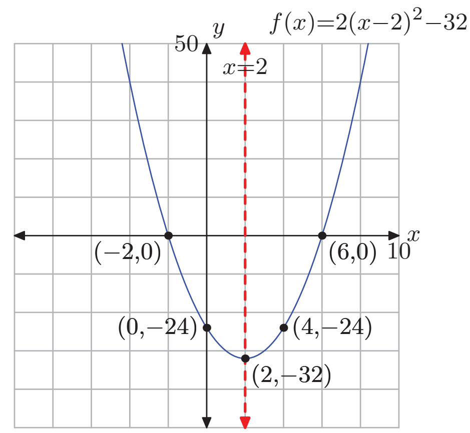
49.



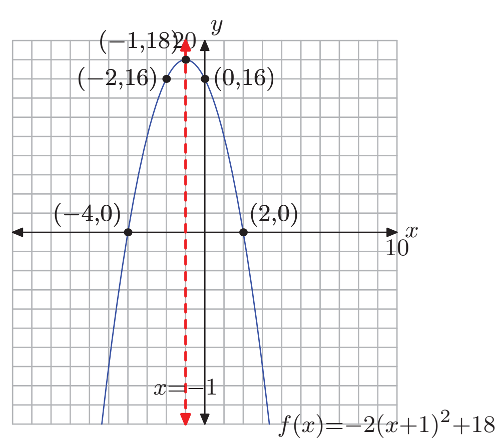
51.



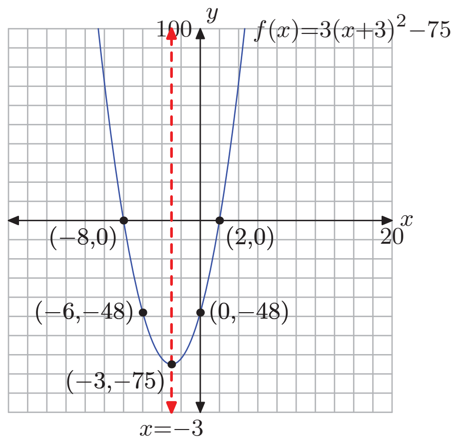
53. Domain $= (-\infty, \infty)$, Range $= [-32, \infty)$



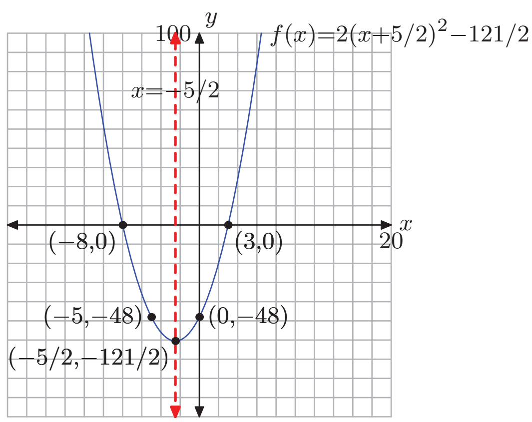
55. Domain $= (-\infty, \infty)$, Range $= (-\infty, 18]$



57. Domain $= (-\infty, \infty)$,
Range $= [-75, \infty)$



59. Domain $= (-\infty, \infty)$,
Range $= [-121/2, \infty)$



61. $-2, 3$

63. $-3, 0$

65. $-3, 0$

2.4 The Quadratic Formula

Consider the general quadratic function

$$f(x) = ax^2 + bx + c.$$

In the previous section, we learned that we can find the zeros of this function by solving the equation

$$f(x) = 0.$$

If we substitute $f(x) = ax^2 + bx + c$, then the resulting equation

$$ax^2 + bx + c = 0 \tag{1}$$

is called a *quadratic equation*. In the previous section, we solved equations of this type by factoring and using the zero product property.

However, it is not always possible to factor the trinomial on the left-hand side of the quadratic **equation (1)** as a product of factors with integer coefficients. For example, consider the quadratic equation

$$2x^2 + 7x - 3 = 0. \tag{2}$$

Comparing $2x^2 + 7x - 3$ with $ax^2 + bx + c$, let's list all integer pairs whose product is $ac = (2)(-3) = -6$.

$1, -6$		$-1, 6$
$2, -3$		$-2, 3$

Not a single one of these integer pairs adds to $b = 7$. Therefore, the quadratic trinomial $2x^2 + 7x - 3$ does not factor over the integers.[11] Consequently, we'll need another method to solve the quadratic **equation (2)**.

The purpose of this section is to develop a formula that will consistently provide solutions of the general quadratic **equation (1)**. However, before we can develop the "Quadratic Formula," we need to lay some groundwork involving the square roots of numbers.

Square Roots

We begin our discussion of square roots by investigating the solutions of the equation $x^2 = a$. Consider the rather simple equation

$$x^2 = 25. \tag{3}$$

Because $(-5)^2 = 25$ and $(5)^2 = 25$, **equation (3)** has two solutions, $x = -5$ or $x = 5$. We usually denote these solutions simultaneously, using a "plus or minus" sign:

[11] This means that the trinomial $2x^2 + 7x - 3$ cannot be expressed as a product of factors with integral (integer) coefficients.

$$x = \pm 5$$

These solutions are called *square roots* of 25. Because there are two solutions, we need a different notation for each. We will denote the positive square root of 25 with the notation $\sqrt{25}$ and the negative square root of 25 with the notation $-\sqrt{25}$. Thus,

$$\sqrt{25} = 5 \qquad \text{and} \qquad -\sqrt{25} = -5.$$

In a similar vein, the equation $x^2 = 36$ has two solutions, $x = \pm\sqrt{36}$, or alternatively, $x = \pm 6$. The notation $\sqrt{36}$ calls for the positive square root, while the notation $-\sqrt{36}$ calls for the negative square root. That is,

$$\sqrt{36} = 6 \qquad \text{and} \qquad -\sqrt{36} = -6.$$

It is not necessary that the right-hand side of the equation $x^2 = a$ be a "perfect square." For example, the equation

$$x^2 = 7 \qquad \text{has solutions} \qquad x = \pm\sqrt{7}. \tag{4}$$

There is no *rational* square root of 7. That is, there is no way to express the square root of 7 in the form p/q, where p and q are integers. Therefore, $\sqrt{7}$ is an example of an irrational number. However, $\sqrt{7}$ is a perfectly valid real number and we're perfectly comfortable leaving our answer in the form shown in **equation (4)**.

However, if an approximation is needed for the square root of 7, we can reason that because 7 lies between 4 and 9, the square root of 7 will lie between 2 and 3. Because 7 is closer to 9 than 4, a reasonable approximation might be[12]

$$\sqrt{7} \approx 2.6.$$

A calculator can provide an even better approximation. For example, our TI83 reports

$$\sqrt{7} \approx 2.645751311.$$

There are two degenerate cases involving the equation $x^2 = a$ that demand our attention.

1. The equation $x^2 = 0$ has only one solution, namely $x = 0$. Thus, $\sqrt{0} = 0$.
2. The equation $x^2 = -4$ has no real solutions.[13] It is not possible to square a real number and get -4. In this situation, we will simply state that "the equation $x^2 = -4$ has no real solutions (no solutions that are real numbers)."

[12] The symbol $\approx$ means "approximately equal to."

[13] It is incorrect to state that the equation $x^2 = -4$ has "no solutions." If we introduce the set of *complex numbers* (a set of numbers introduced in college algebra and trigonometry), then the equation $x^2 = -4$ has two solutions, both of which are complex numbers.

I **Example 5.** *Find all real solutions of the equations* $x^2 = 30$, $x^2 = 0$, *and* $x^2 = -14$.

The solutions follow.

- The equation $x^2 = 30$ has two real solutions, namely $x = \pm\sqrt{30}$.
- The equation $x^2 = 0$ has one real solution, namely $x = 0$.
- The equation $x^2 = -14$ has no real solutions.

Let's try additional examples.

I **Example 6.** *Find all real solutions of the equation* $(x + 2)^2 = 43$.

There are two possibilities for $x + 2$, namely

$$x + 2 = \pm\sqrt{43}.$$

To solve for x, subtract 2 from both sides of this last equation.

$$x = -2 \pm \sqrt{43}.$$

Although this last answer is usually the preferable form of the answer, there are some times when an approximation is needed. So, our TI83 gives the following approximations.

$$-2 - \sqrt{43} \approx -8.557438524 \qquad \text{and} \qquad -2 + \sqrt{43} \approx 4.557438524$$

I **Example 7.** *Find all real solutions of the equation* $(x - 4)^2 = -15$.

If x is a real number, then so is $x - 4$. It's not possible to square the real number $x - 4$ and get -15. Thus, this problem has no real solutions.[14]

Development of the Quadratic Formula

We now have all the groundwork in place to pursue a solution of the general quadratic equation

$$ax^2 + bx + c = 0. \qquad (8)$$

We're going to use a form of "completing the square" to solve this equation for x. Let's begin by subtracting c from both sides of the equation.

$$ax^2 + bx = -c$$

[14] Again, when you study the complex numbers, you will learn that this equation has two complex solutions. Hence, it is important that you do not say that "this problem has no solutions," as that is simply not true. You must say that "this problem has no real solutions."

Next, divide both sides of the equation by a.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Take half of the coefficient of x, as in $(1/2)(b/a) = b/(2a)$. Square this result to get $b^2/(4a^2)$. Add this amount to both sides of the equation.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

On the left we factor the perfect square trinomial. On the right we get a common denominator and add the resulting equivalent fractions.

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Provided the right-hand side of this last equation is positive, we have two real solutions.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

On the right, we take the square root of the top and the bottom of the fraction.[15]

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

To complete the solution, we need only subtract $b/(2a)$ from both sides of the equation.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Although this last answer is a perfectly good solution, we customarily rewrite the solution with a single common denominator.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{9}$$

This last result gives the solution to the general quadratic **equation (8)**. The solution (**9**) is called the *quadratic formula*.

[15] In a later section we will present a more formal approach to the symbolic manipulation of radicals. For now, you can compute $(2/3)^2$ with the calculation $(2/3)(2/3) = 4/9$, or you can simply square numerator and denominator of the fraction, as in $(2/3)^2 = (2^2/3^2) = 4/9$. Conversely, one can take the square root of a fraction by taking the square root of the numerator divided by the square root of the denominator, as in $\sqrt{4/9} = \sqrt{4}/\sqrt{9} = 2/3$.

The Quadratic Formula. The solutions to the *quadratic equation*

$$ax^2 + bx + c = 0 \tag{10}$$

are given by the *quadratic formula*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{11}$$

Although the development of the quadratic formula can be intimidating, in practice its application is quite simple. Let's look at some examples.

I **Example 12.** *Use the quadratic formula to solve the equation*

$$x^2 = 27 - 6x.$$

The first step is to place the equation in the form $ax^2 + bx + c = 0$ by moving every term to one side of the equation,[16] arranging the terms in descending powers of x.

$$x^2 + 6x - 27 = 0$$

Next, compare $x^2 + 6x - 27 = 0$ with the general form of the quadratic equation $ax^2 + bx + c = 0$ and note that $a = 1$, $b = 6$, and $c = -27$. Copy down the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute $a = 1$, $b = 6$, and $c = -27$ and simplify.

$$x = \frac{-(6) \pm \sqrt{(6)^2 - 4(1)(-27)}}{2(1)}$$

$$x = \frac{-6 \pm \sqrt{36 + 108}}{2}$$

$$x = \frac{-6 \pm \sqrt{144}}{2}$$

In this case, 144 is a perfect square. That is, $\sqrt{144} = 12$, so we can continue to simplify.

$$x = \frac{-6 \pm 12}{2}$$

It's important to note that there are *two* real answers, namely

$$x = \frac{-6 - 12}{2} \qquad \text{or} \qquad x = \frac{-6 + 12}{2}.$$

Simplifying,

$$x = -9 \qquad \text{or} \qquad x = 3.$$

[16] We like to say "Make one side equal to zero."

It's interesting to note that this problem could have been solved by factoring. Indeed,

$$\begin{aligned} x^2 + 6x - 27 &= 0 \\ (x-3)(x+9) &= 0, \end{aligned}$$

so the zero product property requires that either $x - 3 = 0$ or $x + 9 = 0$, which leads to $x = 3$ or $x = -9$, answers identical to those found by the quadratic formula.

We'll have more to say about the "discriminant" soon, but it's no coincidence that the quadratic $x^2 + 6x - 27$ factored. Here is the relevant fact.

When the Discriminant is a Perfect Square. In the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

the number under the radical, $b^2 - 4ac$, is called the **discriminant**. When the discriminant is a perfect square, the quadratic function will always factor.

However, it is not always the case that we can factor the given quadratic. Let's look at another example.

I **Example 13.** *Given the quadratic function $f(x) = x^2 - 2x$, find all real solutions of $f(x) = 2$.*

Because $f(x) = x^2 - 2x$, the equation $f(x) = 2$ becomes

$$x^2 - 2x = 2.$$

Set one side of the equation equal to zero by subtracting 2 from both sides of the equation.[17]

$$x^2 - 2x - 2 = 0$$

Compare $x^2 - 2x - 2 = 0$ with the general quadratic equation $ax^2 + bx + c = 0$ and note that $a = 1$, $b = -2$ and $c = -2$. Write down the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Next, substitute $a = 1$, $b = -2$, and $c = -2$. Note the careful use of parentheses.[18]

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)}$$

[17] Note that the quadratic expression on the left-hand side of the resulting equation does not factor over the integers. There are no integer pairs whose product is -2 that sum to -2.

[18] For example, without parentheses, $-2^2 = -4$, whereas with parentheses $(-2)^2 = 4$.

Simplify.

$$x = \frac{2 \pm \sqrt{4+8}}{2}$$
$$x = \frac{2 \pm \sqrt{12}}{2}$$

In this case, 12 is not a perfect square, so we've simplified as much as is possible at this time.[19] However, we can approximate these solutions with the aid of a calculator.

$$x = \frac{2-\sqrt{12}}{2} \approx -0.7320508076 \qquad \text{and} \qquad x = \frac{2+\sqrt{12}}{2} \approx 2.732050808. \tag{14}$$

We will find these approximations useful in what follows.

The equations in **Examples 12** and **13** represent a fundamental shift in our usual technique for solving equations. In the past, we've tried to "isolate" the terms containing x (or whatever unknown we are solving for) on one side of the equation, and all other terms on the other side of the equation. Now, in **Examples 12** and **13**, we find ourselves moving everything to one side of the equation, making one side of the equation equal to zero. This bears some explanation.

Linear or Nonlinear. Let's assume that the unknown we are solving for is x.

- If the highest power of x present in the equation is x to the first power, then the equation is **linear**. Thus, for example, each of the equations

$$2x+3=7, \qquad 3-4x=5x+9, \qquad \text{and} \qquad ax+b=cx+d$$

is linear.
- If there are powers of x higher than x to the first power in the equation, then the equation is **nonlinear**. Thus, for example, each of the equations

$$x^2-4x=9, \qquad x^3=2x+3, \qquad \text{and} \qquad ax^2+bx=cx+d$$

is nonlinear.

The strategy for solving an equation will shift, depending on whether the equation is linear or nonlinear.

[19] In a later chapter on irrational functions, we will take up the topic of simplifying radical expressions. Until then, this form of the final answer will have to suffice.

Solution Strategy—Linear Versus Nonlinear. When solving equations, you must first ask if the equation is linear or nonlinear. Again, let's assume the unknown we wish to solve for is x.

- If the equation is linear, move all terms containing x to one side of the equation, all the remaining terms to the other side of the equation.
- If the equation is nonlinear, move all terms to one side of the equation, making the other side of the equation zero.

Thus, because $ax + b = cx + d$ is linear in x, the first step in solving the equation would be to move all terms containing x to one side of the equation, all other terms to the other side of the equation, as in

$$ax - cx = d - b.$$

On the other hand, the equation $ax^2 + bx = cx + d$ is nonlinear in x, so the first step would be to move all terms to one side of the equation, making the other side of the equation equal to zero, as in

$$ax^2 + bx - cx - d = 0.$$

In **Example 13**, the equation $x^2 - 2x = 2$ is nonlinear in x, so we moved everything to the left-hand side of the equation, making the right-hand side of the equation equal to zero, as in $x^2 - 2x - 2 = 0$. However, it doesn't matter which side you make equal to zero. Suppose instead that you move every term to the right-hand side of the equation, as in

$$0 = -x^2 + 2x + 2.$$

Comparing $0 = -x^2 + 2x + 2$ with general quadratic equation $0 = ax^2 + bx + c$, note that $a = -1$, $b = 2$, and $c = 2$. Write down the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Next, substitute $a = -1$, $b = 2$, and $c = 2$. Again, note the careful use of parentheses.

$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(-1)(2)}}{2(-1)}$$

This leads to two solutions,

$$x = \frac{-2 \pm \sqrt{4 + 8}}{-2} = \frac{-2 \pm \sqrt{12}}{-2}.$$

In **Example 13**, we found the following solutions and their approximations.

$$x = \frac{2 - \sqrt{12}}{2} \approx -0.7320508076 \qquad \text{and} \qquad x = \frac{2 + \sqrt{12}}{2} \approx 2.732050808.$$

It is a fair question to ask if our solutions $x = (-2 \pm \sqrt{12})/(-2)$ are the same. One way to find out is to find decimal approximations of each on our calculator.

$$x = \frac{-2 - \sqrt{12}}{-2} \approx 2.732050808 \qquad \text{and} \qquad x = \frac{-2 + \sqrt{12}}{-2} \approx -0.7320508076.$$

The fact that we get the same decimal approximations should spark confidence that we have the same solutions. However, we can also manipulate the exact forms of our solutions to show that they match the previous forms found in **Example 13**.

Take the two solutions and multiply both numerator and denominator by minus one.

$$\frac{-2 - \sqrt{12}}{-2} = \frac{2 + \sqrt{12}}{2} \qquad \text{and} \qquad \frac{-2 + \sqrt{12}}{-2} = \frac{2 - \sqrt{12}}{2}$$

This shows that our solutions are identical to those found in **Example 13**.

We can do the same negation of numerator and denominator in compact form.

$$\frac{-2 \pm \sqrt{12}}{-2} = \frac{2 \mp \sqrt{12}}{2}$$

Note that this leads to the same two answers, $(2 - \sqrt{12})/2$ and $(2 + \sqrt{12})/2$.

Of the two methods (move all the terms to the left or all the terms to the right), we prefer the approach of **Example 13**. By moving the terms to the left-hand side of the equation, as in $x^2 - 2x - 2 = 0$, the coefficient of x^2 is positive ($a = 1$) and we avoid the minus sign in the denominator produced by the quadratic formula.

Intercepts

In **Example 13**, we used the quadratic formula to find the solutions of $x^2 - 2x - 2 = 0$. These solutions, and their approximations, are shown in **equation (14)**. It is important to make the connection that the solutions in **equation (14)** are the zeros of the quadratic function $g(x) = x^2 - 2x - 2$. The zeros also provide the x-coordinates of the x-intercepts of the graph of g (a parabola). To emphasize this point, let's draw the graph of the parabola having the equation $g(x) = x^2 - 2x - 2$.

First, complete the square to place the quadratic function in vertex form. Take half the middle coefficient and square, as in $[(1/2)(-2)]^2 = 1$; then add and subtract this term so the equation remains balanced.

$$\begin{aligned} g(x) &= x^2 - 2x - 2 \\ g(x) &= x^2 - 2x + 1 - 1 - 2 \end{aligned}$$

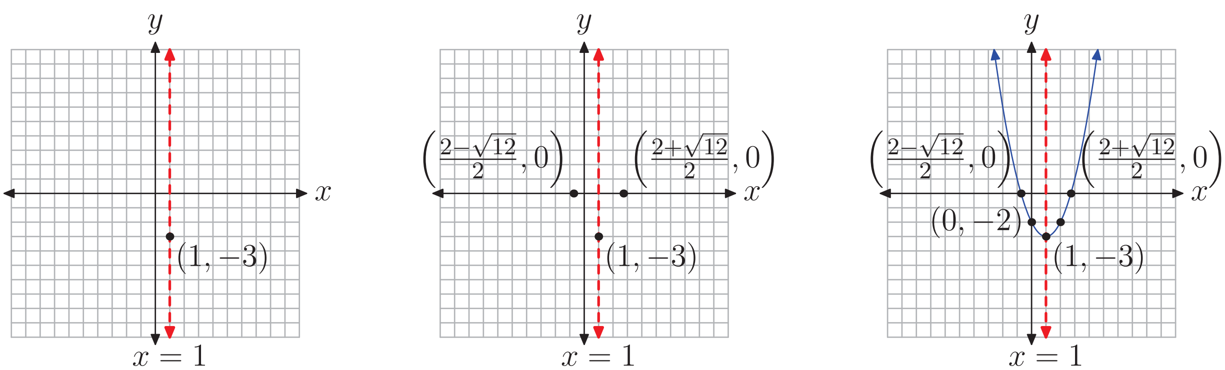
Factor the perfect square trinomial, then combine the constants at the end.

$$g(x) = (x - 1)^2 - 3$$

This is a parabola that opens upward. It is shifted to the right 1 unit and down 3 units. This makes it easy to identify the vertex and draw the axis of symmetry, as shown in **Figure 1**(a).

It will now be apparent why we used our calculator to approximate the solutions in (**14**). These are the x-coordinates of the x-intercepts. One x-intercept is located at approximately $(-0.73, 0)$, the other at approximately $(2.73, 0)$. These approximations are used to plot the location of the intercepts as shown in **Figure 1**(b). However, the actual values of the intercepts are $((2-\sqrt{12})/2, 0)$ and $((2+\sqrt{12})/2, 0)$, and these exact values should be used to annotate the intercepts, as shown in **Figure 1**(b).

Finally, to find the y-intercept, let $x = 0$ in $g(x) = x^2 - 2x - 2$. Thus, $g(0) = -2$ and the y-intercept is $(0, -2)$. The y-intercept and its mirror image across the axis of symmetry are both plotted in **Figure 1**(c), where the final graph of the parabola is also shown.



(a) Plotting the vertex and axis of symmetry.

(b) Adding the x-intercepts provides added accuracy.

(c) Adding the y-intercept and its mirror image provides an excellent final graph.

Figure 1.

We've made an important point and we pause to provide emphasis.

Zeros and Intercepts. Whenever you use the quadratic formula to solve the quadratic equation

$$ax^2 + bx + c = 0,$$

the solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

are the zeros of the quadratic function

$$f(x) = ax^2 + bx + c.$$

The solutions also provide the x-coordinates of the x-intercepts of the graph of f.

We need to discuss one final concept.

The Discriminant

Consider again the quadratic equation $ax^2 + bx + c = 0$ and the solutions (zeros) provided by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The expression under the radical, $b^2 - 4ac$, is called the *discriminant*, which we denote by the letter D. That is, the formula for the discriminant is given by

$$D = b^2 - 4ac.$$

The discriminant is used to determine the nature and number of solutions to the quadratic equation $ax^2+bx+c = 0$. This is done without actually calculating the solutions.

Let's look at three key examples.

I **Example 15.** *Consider the quadratic equation*

$$x^2 - 4x - 4 = 0.$$

Calculate the discriminant and use it to determine the nature and number of the solutions.

Compare $x^2 - 4x - 4 = 0$ with $ax^2 + bx + c = 0$ and note that $a = 1$, $b = -4$, and $c = -4$. The discriminant is given by the calculation

$$D = b^2 - 4ac = (-4)^2 - 4(1)(-4) = 32.$$

Note that the discriminant D is positive; i.e., $D > 0$.

Consider the quadratic function $f(x) = x^2 - 4x - 4$, which can be written in vertex form

$$f(x) = (x - 2)^2 - 8.$$

This is a parabola that opens upward. It is shifted to the right 2 units, then downward 8 units. Therefore, it will cross the x-axis in two locations. Hence, one would expect that the quadratic formula would provide two real solutions (x-intercepts). Indeed,

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-4)}}{2(1)} = \frac{4 \pm \sqrt{32}}{2}.$$

Note that the discriminant, $D = 32$ as calculated above, is the number under the square root. These solutions have approximations

$$x = \frac{4 - \sqrt{32}}{2} \approx -0.8284271247 \qquad \text{and} \qquad x = \frac{4 + \sqrt{32}}{2} \approx 4.828427125,$$

which aid in plotting an accurate graph of $f(x) = (x - 2)^2 - 8$, as shown in **Figure 2**.

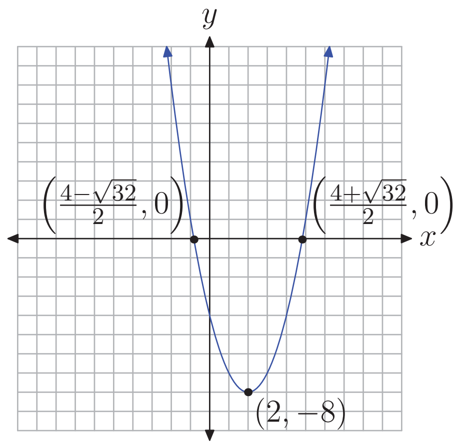


Figure 2. If the discriminant is positive, there are two real x-intercepts.

Thus, if the discriminant is positive, the parabola will have two real x-intercepts.

Next, let's look at an example where the discriminant equals zero.

I **Example 16.** *Consider again the quadratic equation* $ax^2 + bx + c = 0$ *and the solutions (zeros) provided by the quadratic formula*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The expression under the radical, $b^2 - 4ac$*, is called the* discriminant*, which we denote by the letter* D*. That is, the formula for the discriminant is given by*

$$D = b^2 - 4ac.$$

The discriminant is used to determine the nature and number of solutions to the quadratic equation $ax^2+bx+c = 0$*. This is done without actually calculating the solutions. Consider the quadratic equation*

$$x^2 - 4x + 4 = 0.$$

Calculate the discriminant and use it to determine the nature and number of the solutions.

Compare $x^2 - 4x + 4 = 0$ with $ax^2 + bx + c = 0$ and note that $a = 1$, $b = -4$, and $c = 4$. The discriminant is given by the calculation

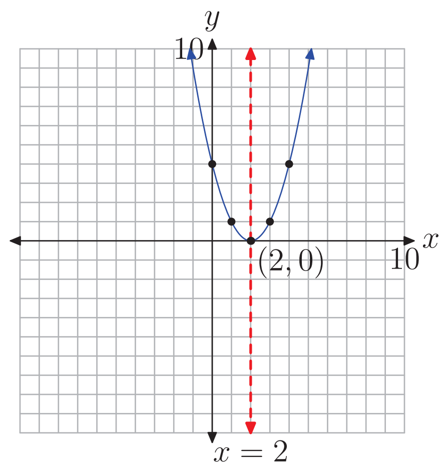
$$D = b^2 - 4ac = (-4)^2 - 4(1)(4) = 0.$$

Note that the discriminant equals zero.

Consider the quadratic function $f(x) = x^2 - 4x + 4$, which can be written in vertex form

$$f(x) = (x-2)^2. \tag{17}$$

This is a parabola that opens upward and is shifted 2 units to the right. Note that there is no vertical shift, so the vertex of the parabola will rest on the x-axis, as shown in **Figure 3**. In this case, we found it necessary to plot two points to the right of the axis of symmetry, then mirror them across the axis of symmetry, in order to get an accurate plot of the parabola.



x	$f(x) = (x-2)^2$
3	1
4	4

Figure 3. At the right is a table of points satisfying $f(x) = (x-2)^2$. These points and their mirror images are seen as solid dots superimposed on the graph of $f(x) = (x-2)^2$ at the left.

Take a closer look at **equation (17)**. If we set $f(x) = 0$ in this equation, then we get $0 = (x-2)^2$. This could be written $0 = (x-2)(x-2)$ and we could say that the solutions are 2 and 2 again. However, mathematicians prefer to say that "2 is a solution of multiplicity 2" or "2 is a double solution."[20] Note how the parabola is tangent to the x-axis at the location of the "double solution." That is, the parabola comes down from positive infinity, touches (but does not cross) the x-axis at $x = 2$, then rises again to positive infinity. Of course, the situation would be reversed in the parabola opened downward, as in $g(x) = -(x-2)^2$, but the graph would still "kiss" the x-axis at the location of the "double solution."

Still, the key thing to note here is the fact that the discriminant $D = 0$ and the parabola has only one x-intercept. That is, the equation $x^2 - 4x + 4 = 0$ has a single real solution.

Next, let's look what happens when the discriminant is negative.

I **Example 18.** *Consider the quadratic equation*

$$x^2 - 4x + 8 = 0.$$

[20] Actually, mathematicians call these "double roots," but we prefer to postpone that language until the chapter on polynomial functions.

Calculate the discriminant and use it to determine the nature and number of the solutions.

Compare $x^2 - 4x + 8 = 0$ with $ax^2 + bx + c = 0$ and note that $a = 1$, $b = -4$, and $c = 8$. The discriminant is given by the calculation

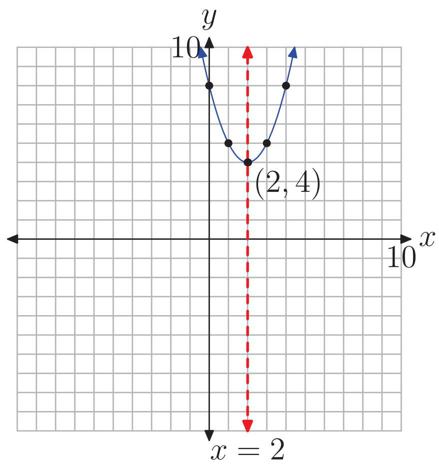
$$D = b^2 - 4ac = (-4)^2 - 4(1)(8) = -16.$$

Note that the discriminant is negative.

Consider the quadratic function $f(x) = x^2 - 4x + 8$, which can be written in vertex form

$$f(x) = (x - 2)^2 + 4.$$

This is a parabola that opens upward. Moreover, it has to be shifted 2 units to the right and 4 units upward, so there can be no x-intercepts, as shown in **Figure 4**. Again, we found it necessary in this example to plot two points to the right of the axis of symmetry, then mirror them, in order to get an accurate plot of the parabola.



x	$f(x) = (x-2)^2+4$
3	5
4	8

Figure 4. At the right is a table of points satisfying $f(x) = (x - 2)^2 + 4$. These points and their mirror images are seen as solid dots superimposed on the graph of $f(x) = (x - 2)^2 + 4$ at the left.

Once again, the key point in this example is the fact that the discriminant is negative and there are no real solutions of the quadratic equation (equivalently, there are no x-intercepts). Let's see what happens if we actually try to find the solutions of $x^2 - 4x + 8 = 0$ using the quadratic formula. Again, $a = 1$, $b = -4$, and $c = 8$, so

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(8)}}{2(1)}.$$

Simplifying,

$$x = \frac{4 \pm \sqrt{-16}}{2}.$$

Again, remember that the number under the square root is the discriminant. In this case the disriminant is -16. It is not possible to square a real number and get -16. Thus, the quadratic equation $x^2 - 4x + 8 = 0$ has no real solutions, as predicted.

Let's summarize the findings in our last three examples.

Summary 19. *Consider the quadratic equation*

$$ax^2 + bx + c = 0.$$

The discriminant is defined as

$$D = b^2 - 4ac.$$

There are three possibilities:

1. *If $D > 0$, then the quadratic equation has two real solutions.*
2. *If $D = 0$, then the quadratic equation has one real solution.*
3. *If $D < 0$, then the quadratic equation has no real solutions.*

This key result is reflected in the graph of the quadratic function.

Summary 20. *Consider the quadratic function*

$$f(x) = ax^2 + bx + c.$$

The graph of this function is a parabola. Three possibilities exist depending upon the value of the discriminant $D = b^2 - 4ac$.

1. *If $D > 0$, the parabola has two x-intercepts.*
2. *If $D = 0$, the parabola has exactly one x-intercept.*
3. *If $D < 0$, the parabola has no x-intercepts.*

2.4 Exercises

In **Exercises 1-8**, find all real solutions of the given equation. Use a calculator to approximate the answers, correct to the nearest hundredth (two decimal places).

1. $x^2 = 36$

2. $x^2 = 81$

3. $x^2 = 17$

4. $x^2 = 13$

5. $x^2 = 0$

6. $x^2 = -18$

7. $x^2 = -12$

8. $x^2 = 3$

In **Exercises 9-16**, find all real solutions of the given equation. Use a calculator to approximate your answers to the nearest hundredth.

9. $(x-1)^2 = 25$

10. $(x+3)^2 = 9$

11. $(x+2)^2 = 0$

12. $(x-3)^2 = -9$

13. $(x+6)^2 = -81$

14. $(x+7)^2 = 10$

15. $(x-8)^2 = 15$

16. $(x+10)^2 = 37$

In **Exercises 17-28**, perform each of the following tasks for the given quadratic function.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. *Remember to draw all lines with a ruler.*
ii. Place the quadratic function in vertex form. Plot the vertex on your coordinate system and label it with its coordinates. Draw the axis of symmetry on your coordinate system and label it with its equation.
iii. Use the quadratic formula to find the x-intercepts of the parabola. Use a calculator to approximate each intercept, correct to the nearest tenth, and use these approximations to plot the x-intercepts on your coordinate system. However, label each x-intercept with its **exact** coordinates.
iv. Plot the y-intercept on your coordinate system and its mirror image across the axis of symmetry and label each with their coordinates.
v. Using all of the information on your coordinate system, draw the graph of the parabola, then label it with the vertex form of the function. Use interval notation to state the domain and range of the quadratic function.

17. $f(x) = x^2 - 4x - 8$

18. $f(x) = x^2 + 6x - 1$

19. $f(x) = x^2 + 6x - 3$

20. $f(x) = x^2 - 8x + 1$

21. $f(x) = -x^2 + 2x + 10$

22. $f(x) = -x^2 - 8x - 8$

23. $f(x) = -x^2 - 8x - 9$

24. $f(x) = -x^2 + 10x - 20$

25. $f(x) = 2x^2 - 20x + 40$

26. $f(x) = 2x^2 - 16x + 12$

27. $f(x) = -2x^2 + 16x + 8$

28. $f(x) = -2x^2 - 24x - 52$

In **Exercises 29-32**, perform each of the following tasks for the given quadratic equation.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. *Remember to draw all lines with a ruler.*
ii. Show that the discriminant is negative.
iii. Use the technique of completing the square to put the quadratic function in vertex form. Plot the vertex on your coordinate system and label it with its coordinates. Draw the axis of symmetry on your coordinate system and label it with its equation.
iv. Plot the y-intercept and its mirror image across the axis of symmetry on your coordinate system and label each with their coordinates.
v. Because the discriminant is negative (did you remember to show that?), there are no x-intercepts. Use the given equation to calculate one additional point, then plot the point and its mirror image across the axis of symmetry and label each with their coordinates.
vi. Using all of the information on your coordinate system, draw the graph of the parabola, then label it with the vertex form of function. Use interval notation to describe the domain and range of the quadratic function.

29. $f(x) = x^2 + 4x + 8$

30. $f(x) = x^2 - 4x + 9$

31. $f(x) = -x^2 + 6x - 11$

32. $f(x) = -x^2 - 8x - 20$

In **Exercises 33-36**, perform each of the following tasks for the given quadratic function.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. *Remember to draw all lines with a ruler.*
ii. Use the discriminant to help determine the value of k so that the graph of the given quadratic function has exactly one x-intercept.
iii. Substitute this value of k back into the given quadratic function, then use the technique of completing the square to put the quadratic function in vertex form. Plot the vertex on your coordinate system and label it with its coordinates. Draw the axis of symmetry on your coordinate system and label it with its equation.
iv. Plot the y-intercept and its mirror image across the axis of symmetry and label each with their coordinates.
v. Use the equation to calculate an additional point on either side of the axis of symmetry, then plot this point and its mirror image across the axis of symmetry and label each with their coordinates.
vi. Using all of the information on your coordinate system, draw the graph of the parabola, then label it with the vertex form of the function. Use

interval notation to describe the domain and range of the quadratic function.

33. $f(x) = x^2 - 4x + 4k$

34. $f(x) = x^2 + 6x + 3k$

35. $f(x) = kx^2 - 16x - 32$

36. $f(x) = kx^2 - 24x + 48$

37. Find all values of k so that the graph of the quadratic function $f(x) = kx^2 - 3x + 5$ has exactly two x-intercepts.

38. Find all values of k so that the graph of the quadratic function $f(x) = 2x^2 + 7x - 4k$ has exactly two x-intercepts.

39. Find all values of k so that the graph of the quadratic function $f(x) = 2x^2 - x + 5k$ has no x-intercepts.

40. Find all values of k so that the graph of the quadratic function $f(x) = kx^2 - 2x - 4$ has no x-intercepts.

In **Exercises 41-50**, find all real solutions, if any, of the equation $f(x) = b$.

41. $f(x) = 63x^2 + 74x - 1;\ b = 8$

42. $f(x) = 64x^2 + 128x + 64;\ b = 0$

43. $f(x) = x^2 - x - 5;\ b = 2$

44. $f(x) = 5x^2 - 5x;\ b = 3$

45. $f(x) = 4x^2 + 4x - 1;\ b = -2$

46. $f(x) = 2x^2 - 9x - 3;\ b = -1$

47. $f(x) = 2x^2 + 4x + 6;\ b = 0$

48. $f(x) = 24x^2 - 54x + 27;\ b = 0$

49. $f(x) = -3x^2 + 2x - 13;\ b = -5$

50. $f(x) = x^2 - 5x - 7;\ b = 0$

In **Exercises 51-60**, find all real solutions, if any, of the quadratic equation.

51. $-2x^2 + 7 = -3x$

52. $-x^2 = -9x + 7$

53. $x^2 - 2 = -3x$

54. $81x^2 = -162x - 81$

55. $9x^2 + 81 = -54x$

56. $-30x^2 - 28 = -62x$

57. $-x^2 + 6 = 7x$

58. $-8x^2 = 4x + 2$

59. $4x^2 + 3 = -x$

60. $27x^2 = -66x + 16$

In **Exercises 61-66**, find all of the x-intercepts, if any, of the given function.

61. $f(x) = -4x^2 - 4x - 5$

62. $f(x) = 49x^2 - 28x + 4$

63. $f(x) = -56x^2 + 47x + 18$

64. $f(x) = 24x^2 + 34x + 12$

65. $f(x) = 36x^2 + 96x + 64$

66. $f(x) = 5x^2 + 2x + 3$

In **Exercises 67-74**, determine the number of real solutions of the equation.

67. $9x^2 + 6x + 1 = 0$

68. $7x^2 - 12x + 7 = 0$

69. $-6x^2 + 4x - 7 = 0$

70. $-8x^2 + 11x - 4 = 0$

71. $-5x^2 - 10x - 5 = 0$

72. $6x^2 + 11x + 2 = 0$

73. $-7x^2 - 4x + 5 = 0$

74. $6x^2 + 10x + 4 = 0$

2.4 Answers

1. $x = \pm 6$

3. $x = \pm\sqrt{17} = \pm 4.12$

5. $x = 0$

7. No real solutions.

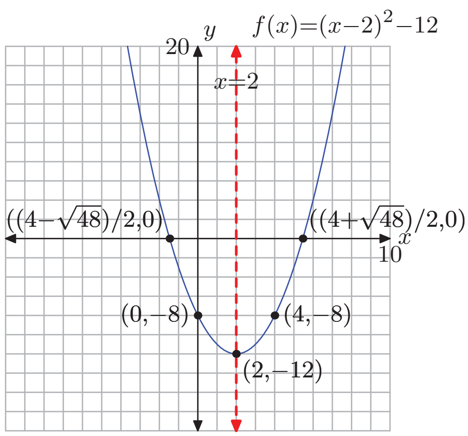
9. $x = -4$ or $x = 6$

11. $x = -2$

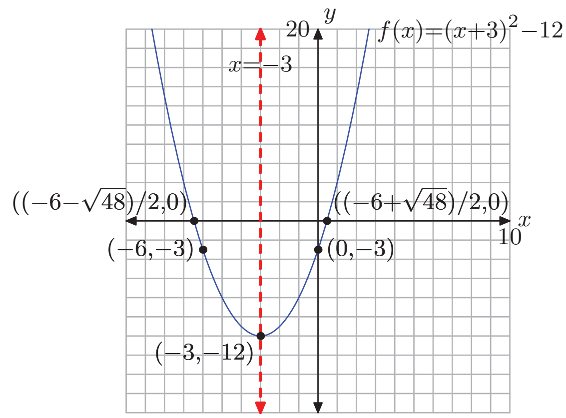
13. No real solutions.

15. $x = 8 \pm \sqrt{15} \approx 4.13,\ 11.87$

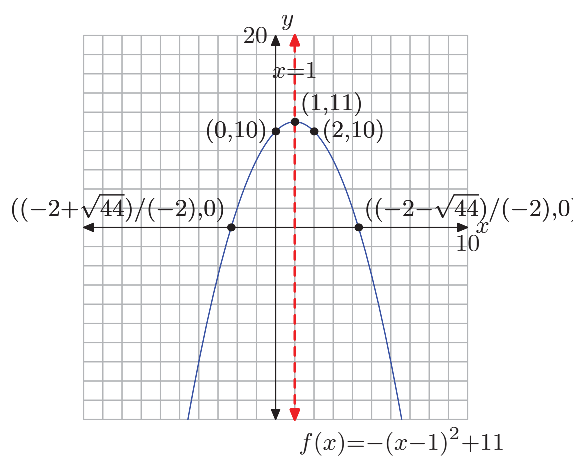
17. Domain $= (-\infty, \infty)$,
Range $= [-12, \infty)$



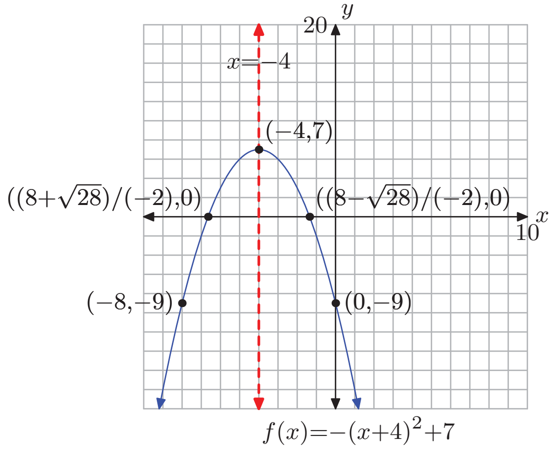
19. Domain $= (-\infty, \infty)$,
Range $= [-12, \infty)$



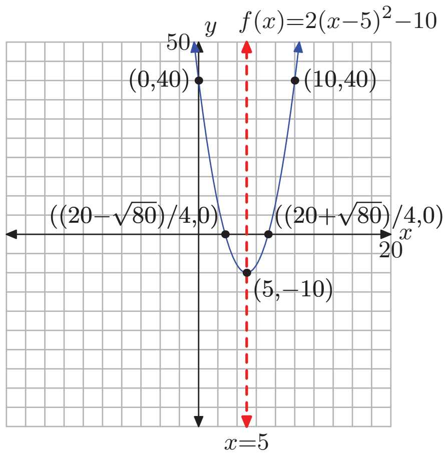
21. Domain $= (-\infty, \infty)$,
Range $= (-\infty, 11]$



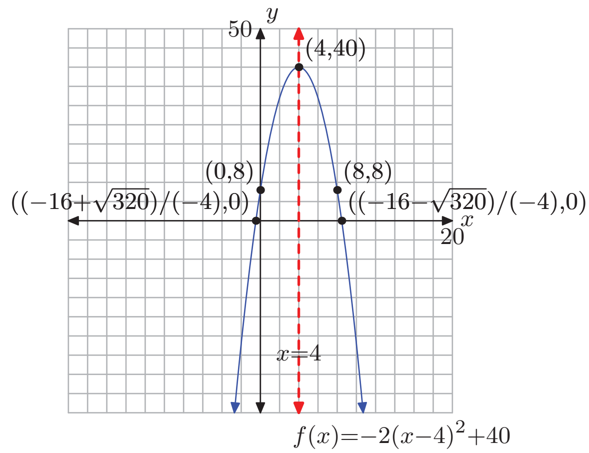
23. Domain $= (-\infty, \infty)$, Range $= (-\infty, 7]$



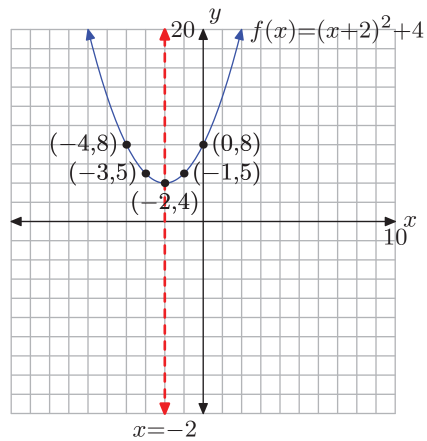
25. Domain $= (-\infty, \infty)$, Range $= [-10, \infty)$



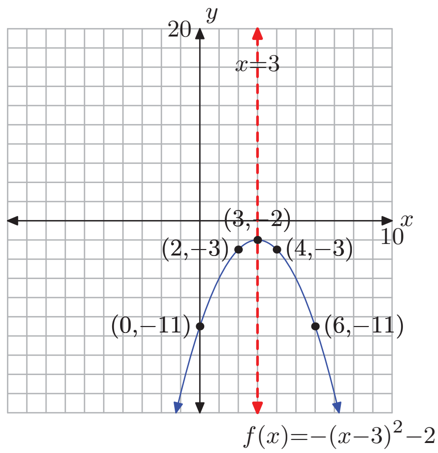
27. Domain $= (-\infty, \infty)$, Range $= (-\infty, 40]$



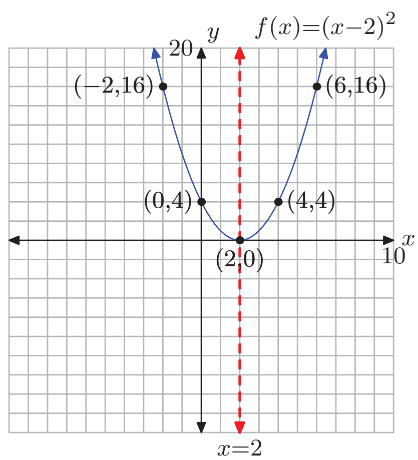
29. Domain $= (-\infty, \infty)$, Range $= [4, \infty)$



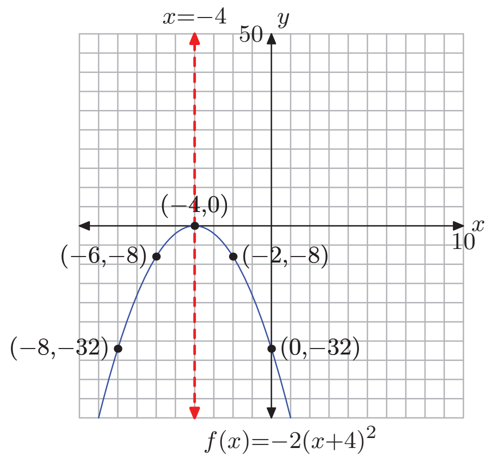
31. Domain $= (-\infty, \infty)$, Range $= (-\infty, -2]$



33. $k = 1$, Domain $= (-\infty, \infty)$, Range $= [0, \infty)$



35. $k = -2$, Domain $= (-\infty, \infty)$, Range $= (-\infty, 0]$



37. $\{k : k < 9/20\}$

39. $\{k : k > 1/40\}$

41. $-\frac{9}{7}, \frac{1}{9}$

43. $\frac{1+\sqrt{29}}{2}, \frac{1-\sqrt{29}}{2}$

45. $-\frac{1}{2}$

47. no real solutions

49. no real solutions

51. $\frac{3-\sqrt{65}}{4}, \frac{3+\sqrt{65}}{4}$

53. $-\frac{3-\sqrt{17}}{2}, -\frac{3+\sqrt{17}}{2}$

55. -3

57. $-\frac{7+\sqrt{73}}{2}, -\frac{7-\sqrt{73}}{2}$

59. no real solutions

61. no x-intercepts

63. $(\frac{9}{8}, 0)$, $(-\frac{2}{7}, 0)$

65. $(-\frac{4}{3}, 0)$

67. 1

69. 0

71. 1

73. 2

2.5 Complex Numbers

Number haters are gonna hate. People have never liked the idea that there are types of numbers that no one has thought of before, and they usually assume that such numbers couldn't possible be useful. The number zero was controversial (how could there be a number to represent nothing?!?). Negative numbers were equally controversial. In Ancient Greece, where religion was tied closely to mathematics, some people were jailed or killed for speaking the heresy that $\sqrt{2}$ could not be obtained by dividing two integers! No one is really upset by any of these things today. All of these numbers are accepted as part of the set of *"real numbers"* (which may be thought of as all of the numbers on the number line), and solving a lot of real life problems would be pretty hard without them.

So, are the *real numbers* the only numbers there are? Nope. There is a larger system of numbers that includes all of the real numbers *and* infinitely more numbers. This is called the set of *complex numbers* (despite their name, using them is not very difficult), and without this number system, it would be very difficult to design the electronic devices that we take for granted in our lives. So, don't hate. Here we go ...

What is missing from the *real numbers* are square roots of *negative numbers*. If we are limited to the *real numbers*, then $\sqrt{\textit{any negative number}}$ is *undefined* since any number on the number line multiplied by itself is either positive or zero, but never negative. For example, no real number multiplied by itself gives you -1. The only possible candidate is -1, but $(-1)(-1) = 1$, so it doesn't work.

Definition. In the *complex numbers* we defined a new number i such that $i^2 = -1$. This means that $i = \sqrt{-1}$. We also include all positive and negative multiples of i such as $2i, 3i, 4i, -i, -2i, -3i, \ldots$ Including these numbers means that all negative numbers have square roots.

Facts about i

1. They aren't crazy. Get over it. (i makes your iPhone possible)!
2. $i^2 = -1$
3. $i = \sqrt{-1}$
4. If you see a square root of a negative, you must use the following rule to "put an i out" <u>before</u> using any other rules!

$$\sqrt{-k} = \sqrt{k} \cdot i \qquad \text{(for exmple } \sqrt{-9} = \sqrt{9} \cdot i = 3i)$$

5. Adding and Subtracting *complex numbers* is really just combining like terms. Terms with i in them have a common factor.

$$\begin{aligned} 5i + 2i &= 7i \quad (\textit{just like } 5x + 2x = 7x). \\ 10i - 6i &= 4i \quad (\textit{just like } 10x - 6x = 4x). \\ 3i + 4 & \quad (\textit{not like terms, cannot be simplified.} \\ & \qquad \text{Same with } 3x + 4) \end{aligned}$$

▶ **Example 1.** *Simplify* $\sqrt{-9} \cdot \sqrt{-4}$.

$\sqrt{-9} \cdot \sqrt{-4}$ remember to "put an i out" before using any other rules

$= \sqrt{9} \cdot i \cdot \sqrt{4}i$

$= 3 \cdot 2 \cdot i \cdot i$

$= 6i^2$

$= 6(-1)$

$= -6$

Warning

Note: if you had forgotten to "put an i out" at the beginning you would get an incorrect answer!

▶ **Example 2.** *Simplify* $(4i + 7) - (2i - 5)$.

$(4i + 7) - (2i - 5)$	distribute the negative
$= 4i + 7 - 2i + 5$	combine like terms
$= 2i + 12$	not like terms, so can't simplify further.

▶ **Example 3.** *Solve the equation* $x^2 + 4 = 0$.

$x^2 + 4 = 0$	can't factor, so use square root property
$x^2 = -4$	
$x = \pm\sqrt{-4}$	put an i out and simplify
$x = \pm\sqrt{4} \cdot i$	
$x = \pm 2i$	

▶ **Example 4.** *Solve the equation* $x^2 + 2x + 5 = 0$.

$x^2 + 2x + 5 = 0$	use quadratic formula
$x = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)}$	
$x = \dfrac{-2 \pm \sqrt{-16}}{2}$	put an i out and simplify
$x = \dfrac{-2 \pm \sqrt{16} \cdot i}{2}$	put an i out and simplify
$x = \dfrac{-2 \pm 4i}{2}$	ALL terms have a factor of 2, so we can cancel
$x = \dfrac{2 \cdot (-1 \pm 2i)}{2 \cdot (1)}$	
$x = -1 \pm 2i$	no more simplification is possible (not like terms)

2.5 Exercises

In Exercises 1-10, simplify.

1. $\sqrt{-9}$

2. $\sqrt{-100}$

3. $-\sqrt{-50}$

4. $\sqrt{-200}$

5. $\sqrt{-\frac{13}{4}}$

6. $\sqrt{-\frac{11}{13}}$

7. $\sqrt{-12}$

8. $\sqrt{140-180}$

9. $\dfrac{15-5\sqrt{-11}}{5}$

10. $\dfrac{6+\sqrt{-24}}{4}$

In Exercises 11-17, find all complex solutions.

11. $x^2 = -25$

12. $x^2 + 16 = 0$

13. $4x^2 + 28 = 0$

14. $11x^2 = -36$

15. $(x-9)^2 = -343$

16. $\left(x+\frac{5}{2}\right)^2 = -\frac{6}{4}$

17. $-2(y-1)^2 + 10 = 59$

In Exercises 18-19, find all complex solutions by completing the square.

18. $x^2 - 14x + 58 = 0$

19. $x^2 + x + 2 = 0$

In Exercises 20-25, find all complex solutions by using the quadratic formula.

20. $x^2 - 14x + 53 = 0$

21. $x^2 + x + 8 = 0$

22. $-16x^2 = -3x + 1$

23. $6x^2 - 7x = -4$

24. $2p^2 - \frac{5}{2}p + 3 = 0$

25. $x^2 + 2x + 6 = 0$

In Exercises 26-32, solve by the method of your choice.

26. $9x^2 = -49$

27. $y^2 + 12y = -575$

28. $(x+4)^2 = -125$

29. $-2(y-1)^2 + 14 = 95$

30. $3x^2 + 6x = -10$

31. $81x^2 + 1 = 0$

32. $-4x^2 + 4x = 9$

2.5 Answers

1. $3i$

3. $-5i\sqrt{2}$

5. $\frac{i\sqrt{13}}{2}$

7. $2i\sqrt{3}$

9. $3 - i\sqrt{11}$

11. $\pm 5i$

13. $\pm i\sqrt{7}$

15. $9 \pm 7i\sqrt{7}$

17. $\frac{2 \pm 7i\sqrt{2}}{2}$

19. $\frac{-1 \pm i\sqrt{7}}{2}$

21. $\frac{-1 \pm i\sqrt{31}}{2}$

23. $\frac{7 \pm i\sqrt{47}}{12}$

25. $-1 \pm i\sqrt{5}$

27. $-6 \pm 7i\sqrt{11}$

29. $\frac{2 \pm 9i\sqrt{2}}{2}$

31. $\pm\frac{1}{9}i$

2.6 Optimization

In this section we will explore the science of optimization. Suppose that you are trying to find a pair of numbers with a fixed sum so that the product of the two numbers is a maximum. This is an example of an optimization problem. However, optimization is not limited to finding a maximum. For example, consider the manufacturer who would like to minimize his costs based on certain criteria. This is another example of an optimization problem. As you can see, optimization can encompass finding either a maximum or a minimum.

Optimization can be applied to a broad family of different functions. However, in this section, we will concentrate on finding the maximums and minimums of quadratic functions. There is a large body of real-life applications that can be modeled by quadratic functions, so we will find that this is an excellent entry point into the study of optimization.

Finding the Maximum or Minimum of a Quadratic Function

Consider the quadratic function

$$f(x) = -x^2 + 4x + 2.$$

Let's complete the square to place this quadratic function in vertex form. First, factor out a minus sign.

$$f(x) = -\left[x^2 - 4x - 2\right]$$

Take half of the coefficient of x and square, as in $[(1/2)(-4)]^2 = 4$. Add and subtract this amount to keep the equation balanced.

$$f(x) = -\left[x^2 - 4x + 4 - 4 - 2\right]$$

Factor the perfect square trinomial, combine the constants at the end, and then redistribute the minus sign to place the quadratic function in vertex form.

$$f(x) = -\left[(x-2)^2 - 6\right]$$
$$f(x) = -(x-2)^2 + 6$$

This is a parabola that opens downward, has been shifted 2 units to the right and 6 units upward. This places the vertex of the parabola at $(2, 6)$, as shown in **Figure 1**. Note that the maximum function value (y-value) occurs at the vertex of the parabola. A mathematician would say that the function "attains a maximum value of 6 at x equals 2."

Note that 6 is greater than or equal to any other y-value (function value) that occurs on the parabola. This gives rise to the following definition.

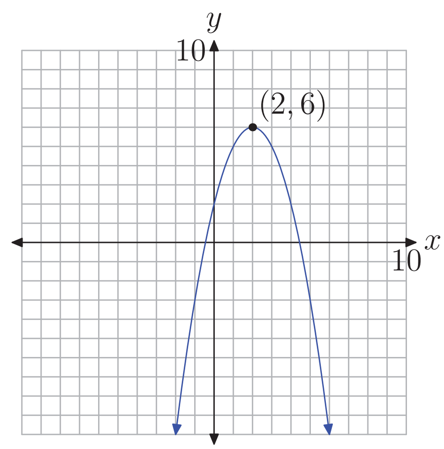


Figure 1. The maximum value of the function, 6, occurs at the vertex of the parabola, $(2, 6)$.

Definition 1. *Let c be in the domain of f. The function f is said to achieve a maximum at $x = c$ if $f(c) \geq f(x)$ for all x in the domain of f.*

Next, let's look at a quadratic function that attains a minimum on its domain.

I **Example 2.** *Find the minimum value of the quadratic function defined by the equation*

$$f(x) = 2x^2 + 12x + 12.$$

Factor out a 2.

$$f(x) = 2\left[x^2 + 6x + 6\right] \tag{3}$$

Take half of the coefficient of x and square, as in $[(1/2)(6)]^2 = 9$. Add and subtract this amount to keep the equation balanced.

$$f(x) = 2\left[x^2 + 6x + 9 - 9 + 6\right]$$

Factor the trinomial and combine the constants, and then redistribute the 2 in the next step.

$$f(x) = 2\left[(x+3)^2 - 3\right]$$
$$f(x) = 2(x+3)^2 - 6$$

The graph is a parabola that opens upward, shifted 3 units to the left and 6 units downward. This places the vertex at $(-3, -6)$, as shown in **Figure 2**. Note that the minimum function value (y-value) occurs at the vertex of the parabola. A mathematician would say that the function "attains a minimum value of -6 at x equals -3.

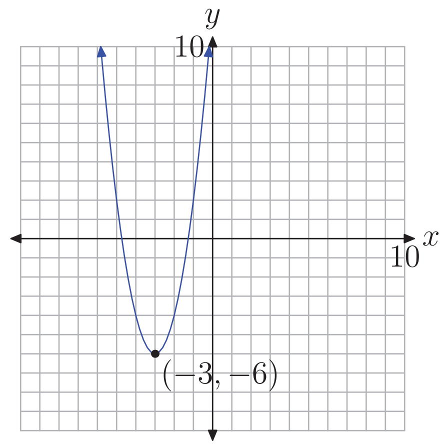


Figure 2. The minimum value of the function, -6, occurs at the vertex of the parabola, $(-3, -6)$.

Note that -6 is less than or equal to any other y-value (function value) that occurs on the parabola.

This last example gives rise to the following definition.

Definition 4. *Let c be in the domain of f. The function f is said to achieve a minimum at $x = c$ if $f(c) \leq f(x)$ for all x in the domain of f.*

A Shortcut for the Vertex

It should now be clear that the vertex of the parabola plays a crucial role when optimizing a quadratic function. We also know that we can complete the square to find the coordinates of the vertex. However, it would be nice if we had a quicker way of finding the coordinates of the vertex. Let's look at the general quadratic function

$$y = ax^2 + bx + c$$

and complete the square to find the coordinates of the vertex. First, factor out the a.

$$y = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

Take half of the coefficient of x and square, as in $[(1/2)(b/a)]^2 = [b/(2a)]^2 = b^2/(4a^2)$. Add and subtract this amount to keep the equation balanced.

$$y = a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right]$$

Factor the perfect square trinomial and make equivalent fractions for the constant terms with a common denominator.

$$y = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{4ac}{4a^2}\right]$$
$$y = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$$

Finally, redistribute that a. Note how multiplying by a cancels one a in the denominator of the constant term.

$$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

Now, here's the key idea. The results depend upon the values of a, b, and c, but it should be clear that the coordinates of the vertex are

$$\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right).$$

The y-value of the vertex is a bit hard to memorize, but the x-value of the vertex is easy to memorize.

Vertex Shortcut. Given the parabola represented by the quadratic function

$$y = ax^2 + bx + c,$$

the x-coordinate of the vertex is given by the formula

$$x_{\text{vertex}} = -\frac{b}{2a}.$$

Let's test this with the quadratic function given in **Example 2**

I **Example 5.** *Use the formula $x_{\text{vertex}} = -b/(2a)$ to find the x-coordinate of the vertex of the parabola represented by the quadratic function in* **Example 2**.

In **Example 2**, the quadratic function was represented by the equation

$$f(x) = 2x^2 + 12x + 12.$$

In vertex form

$$f(x) = 2(x + 3)^2 - 6,$$

the coordinates of the vertex were easily seen to be $(-3, -6)$ (see **Figure 2**). Let's see what the new formula for the x-coordinate of the vertex reveals.

As usual, compare $f(x) = 2x^2 + 12x + 12$ with $f(x) = ax^2 + bx + c$ and note that $a = 2$, $b = 12$ and $c = 12$. Thus, the x-coordinate of the vertex is given by

$$x_{\text{vertex}} = -\frac{b}{2a} = -\frac{12}{2(2)} = -3.$$

Note that this agrees with the previous result (see **Figure 2**). We could find the y-coordinate of the vertex with

$$y_{\text{vertex}} = \frac{4ac - b^2}{4a} = \frac{4(2)(12) - (12)^2}{4(2)} = \frac{-48}{8} = -6,$$

but we find this formula for the y-coordinate of the vertex a bit hard to memorize. We find it easier to do the following. Since we know the x-coordinate of the vertex is $x = -3$, we can find the y-coordinate of the vertex by simply substituting $x = -3$ in the equation of the parabola. That is, with $f(x) = 2x^2 + 12x - 12$,

$$f(-3) = 2(-3)^2 + 12(-3) + 12 = -6.$$

Let's highlight this last technique.

Finding the y-coordinate of the Vertex. Given the parabola represented by the quadratic function

$$f(x) = ax^2 + bx + c,$$

we've seen that the x-coordinate of the vertex is given by $x = -b/(2a)$. To find the y-coordinate of the vertex, it is probably easiest to evaluate the function at $x = -b/(2a)$. That is, the y-coordinate of the vertex is given by

$$y_{\text{vertex}} = f\left(-\frac{b}{2a}\right).$$

Let's look at another example.

I **Example 6.** *Consider the parabola having equation*

$$f(x) = -2x^2 + 3x - 8.$$

Find the coordinates of the vertex.

First, use the new formula to find the x-coordinate of the vertex.

$$x_{\text{vertex}} = -\frac{b}{2a} = -\frac{3}{2(-2)} = \frac{3}{4}.$$

Next, substitute $x = 3/4$ to find the corresponding y-coordinate.

$$\begin{aligned} f\left(\frac{3}{4}\right) &= -2\left(\frac{3}{4}\right)^2 + 3\left(\frac{3}{4}\right) - 8 \\ &= -2\left(\frac{9}{16}\right) + \frac{9}{4} - 8 \\ &= -\frac{9}{8} + \frac{18}{8} - \frac{64}{8} \\ &= -\frac{55}{8} \end{aligned}$$

Thus, the coordinates of the vertex are $(3/4, -55/8)$.

Applications

We're now in a position to do some applications of optimization. Let's start with an easy example.

I **Example 7.** *Find two real numbers x and y that sum to 50 and that have a product that is a maximum.*

Before we apply the theory of the previous examples, let's just play with the numbers a bit to get a feel for what we are being asked to do. We need to find two numbers that sum to 50, so let's start with $x = 5$ and $y = 45$. Clearly, the sum of these two numbers is 50. On the other hand, their product is $xy = (5)(45) = 225$. Let's place this result in a table.

x	y	xy
5	45	225

For a second guess, select $x = 10$ and $y = 40$. The sum of these two numbers is 50 and their product is $xy = 400$. For a third guess, select $x = 20$ and $y = 30$. The sum of these two numbers is 50 and their product is $xy = 600$. Let's add these results to our table.

x	y	xy
5	45	225
10	40	400
20	30	600

Thus far, the best pair is $x = 20$ and $y = 30$, because their product is the maximum in the table above. But is there another pair with a larger product? Remember our goal is to find a pair with a product that is a maximum. That is, our pair must have a product larger than any other pair. Can you find a pair that has a product larger than 600?

Now that we have a feel for what we are being asked to do (find two numbers that sum to 50 and that have a product that is a maximum), let's try an approach that is more abstract than the "guess and check" approach of our tables. Our first constraint is the fact that the sum of the numbers x and y must be 50. We can model this constraint with the equation

$$x + y = 50. \tag{8}$$

We're being asked to maximize the product. Thus, you want to find a formula for the product. Let's let P represent the product of x and y and write

$$P = xy. \tag{9}$$

Note that P is a function of **two** variables x and y. However, all of our functions in this course have thus far been a function of a single variable. So, how can we get rid of one of the variables? Simple, first solve **equation (8)** for y.

$$\begin{aligned} x + y &= 50 \\ y &= 50 - x \end{aligned} \tag{10}$$

Now, substitute **equation (10)** into the product in **equation (9)**.

$$P = x(50 - x),$$

or, equivalently,

$$P = -x^2 + 50x. \tag{11}$$

Note that P is now a function of a single variable x. Note further that the function defined by **equation (11)** is quadratic. If we compare $P = -x^2 + 50x$ with the general form $P = ax^2 + bx + c$, note that $a = -1$ and $b = 50$ (we have no need of the fact that $c = 0$). Therefore, if we plot P versus x, the graph is a parabola that opens downward (see **Figure 3**) and the maximum value of P will occur at the vertex. The x-coordinate of the vertex is found with

$$x_{\text{vertex}} = -\frac{b}{2a} = -\frac{50}{2(-1)} = 25.$$

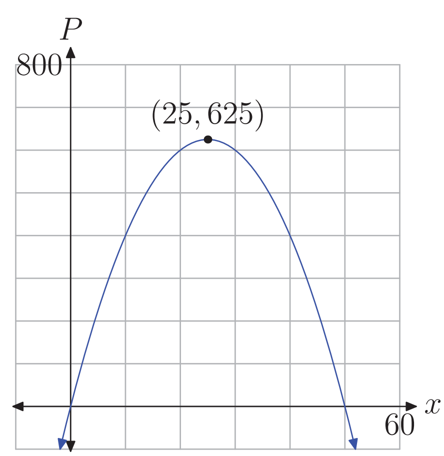


Figure 3. The maximum product P=625 occurs at the vertex of the parabola, $(25, 625)$.

Thus, our first number is $x = 25$. We can find the second number y by substituting $x = 25$ in **equation (10)**.

$$y = 50 - x = 50 - 25 = 25.$$

Note that the sum of x and y is $x + y = 25 + 25 = 50$. There are two ways that we can find their product. Since we now know the numbers x and y, we can multiply to find $P = xy = (25)(25) = 625$. Alternatively, we could substitute $x = 25$ in **equation (11)** to get

$$P = -x^2 + 50x = -(25)^2 + 50(25) = -625 + 1250 = 625.$$

When you compare this result with our experimental tables, things come together. We've found two numbers x and y that sum to 50 with a product that is a maximum. No other numbers that sum to 50 have a larger product.

Our little formula $x_{\text{vertex}} = -b/(2a)$ has proven to be a powerful ally. Let's try another example.

I **Example 12.** *Find two real numbers with a difference of 8 such that the sum of the squares of the two numbers is a minimum.*

Let's begin by letting x and y represent the numbers we seek. Next, let's play a bit as we did in the previous example. Try $x = 9$ and $y = 1$. The difference of these two numbers is certainly 8. The sum of the squares of these two numbers is $S = 9^2 + 1^2 = 82$. Let's put this result in tabular form.

x	y	$S = x^2 + y^2$
9	1	82

For a second guess, select $x = 8$ and $y = 0$. The difference is $x - y = 8 - 0 = 8$, but this time the sum of the squares is $S = 8^2 + 0^2 = 64$. For a third guess, try $x = 7$ and $y = -1$. Again, the difference is $x - y = 7 - (-1) = 8$, but the sum of the squares is now $S = 7^2 + (-1)^2 = 50$. Let's add these results to our table.

x	y	$S = x^2 + y^2$
9	1	82
8	0	64
7	−1	50

Thus far, the pair that minimizes the sum of the squares is $x = 7$ and $y = -1$. However, could there be another pair with a difference of 8 and the sum of the squares is smaller than 50? Experiment further to see if you can best the current minimum of 50.

Let's try an analytical approach. Our first constraint is the fact that the difference of the two numbers must equal 8. This is easily expressed as

$$x - y = 8. \tag{13}$$

Next, we're asked to minimize the sum of the squares of the two numbers. This requires that we find a formula for the sum of the squares. Let S represent the sum of the squares of x and y. Thus,

$$S = x^2 + y^2. \tag{14}$$

Note that S is a function of two variables. We can eliminate one of the variables by solving **equation (13)** for x,

$$x = y + 8, \tag{15}$$

then substituting this result in **equation (14)**.

$$S = (y + 8)^2 + y^2.$$

Expand and simplify.

$$S = 2y^2 + 16y + 64 \tag{16}$$

Compare $S = 2y^2 + 16y + 64$ with the general quadratic $S = ay^2 + by + c$ and note that $a = 2$ and $b = 16$. Thus, the plot of S versus y will be a parabola that opens upward (see **Figure 4**) and the minimum value of S will occur at the vertex. The y-coordinate of the vertex[27] is found with

$$y_{\text{vertex}} = -\frac{b}{2a} = -\frac{16}{2(2)} = -4.$$

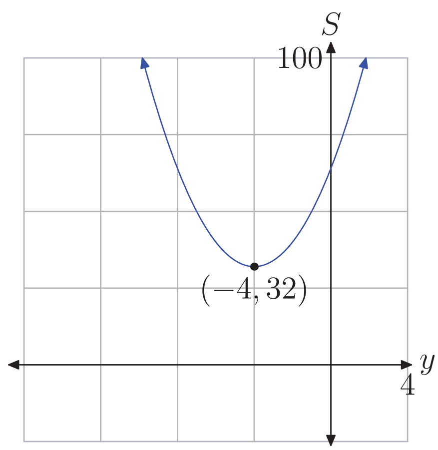


Figure 4. Plotting the sum of the squares S versus y. The minimum S, 32, occurs at the vertex, $(-4, 32)$.

Thus, the first number we seek is $y = -4$. We can find the second number by substituting $y = -4$ in **equation (15)**.

[27] Because we've plotted S versus y, the horizontal axis is labeled y. Thus, y has taken the usual role of x. That's why we write $y_{\text{vertex}} = -b/(2a)$ instead of $x_{\text{vertex}} = -b/(2a)$ in this example.

$$x = y + 8 = (-4) + 8 = 4.$$

Hence, the numbers we seek are $x = 4$ and $y = -4$. Note that the difference of these two numbers is $x - y = 4 - (-4) = 8$ and the sum of their squares is $S = (4)^2 + (-4)^2 = 32$, which is smaller than the best result found in our tabular experiment above. Indeed, our work show that this is the smallest possible value of S.

Alternatively, you can find S by substituting $y = -4$ in **equation (16)**. We'll leave it to our readers to verify that this also gives a minimum value of $S = 32$.

Let's look at another application.

I **Example 17.** *Mary wants to fence a rectangular garden to keep the deer from eating her fruit and vegetables. One side of her garden abuts her shed wall so she will not need to fence that side. However, she also wants to use material to separate the rectangular garden in two sections (see* **Figure 5***). She can afford to buy 80 total feet of fencing to use for the perimeter and the section dividing the rectangular garden. What dimensions will maximize the total area of the rectangular garden?*

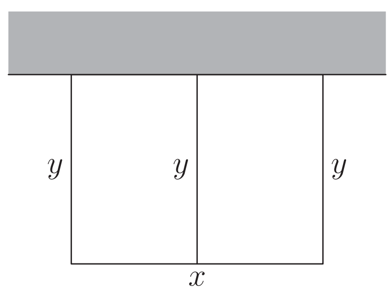


Figure 5. Mary's rectangular garden needs fencing on three sides and also for the fence to divide the garden.

Again, before we take an algebraic approach, let's just experiment. Note that we've labeled the width with the letter x and the height with the letter y in our sketch of the garden in **Figure 5**.

There is a total of 80 feet of fence material. Suppose that we let $y = 5$ ft. Because there are three sides of length $y = 5$ ft, we've used 15 feet of material. That leaves 65 feet of material which will be used to fence the width of the garden. That is, the width is $x = 65$ ft. Thus, the dimensions of the garden are $x = 65$ ft by $y = 5$ ft. The area equals the product of these two measures, so $A = 325\,\text{ft}^2$. Let's put this result into a table.

x	y	$A = xy$
65 ft	5 ft	$325\,\text{ft}^2$

Suppose instead that we let the height be $y = 10$ ft. Again, there are three sections with this length, so this will take 30 ft of material. That leaves 50 ft of material, so the width $x = 50$ ft. The area is the product of these two measures, so $A = 500\,\text{ft}^2$. As a third experiment, let the height $y = 15$ ft. Subtracting three of these lengths from 80 ft, we see that the width $x = 35$ ft. The area is the product of these measures, so $A = 525\,\text{ft}^2$. Let's add these last two number experiments to our table.

x	y	$A = xy$
65 ft	5 ft	$325\,\text{ft}^2$
50 ft	10 ft	$500\,\text{ft}^2$
35 ft	15 ft	$525\,\text{ft}^2$

At this point, the last set of dimensions yields the maximum area, but is it possible that another choice of x and y will yield a larger area? Experiment further with numbers of your choice to see if you can find dimensions that will yield an area larger than the current maximum in the table, namely $525\,\text{ft}^2$.

Let's now call on what we've learned in this section to attack this model. First, we're constrained by the amount of material we have for the job, a total of 80 ft of fencing. This constraint requires that 3 times the height of the garden, added to the width of the garden, should equal the available amount of fencing material. In symbols,

$$x + 3y = 80. \tag{18}$$

We're asked to maximize the area, so we focus our efforts on finding a formula for the area of the rectangular garden. Because the area A of the rectangular garden is the product of the width and the height,

$$A = xy. \tag{19}$$

We now have a formula for the area of the rectangular garden, but unfortunately we have the area A as a function of **two** variables. We need to eliminate one or the other of these variables. This is easily done by solving **equation (18)** for x.

$$x = 80 - 3y \tag{20}$$

Next, substitute this result in **equation (19)** to get

$$A = (80 - 3y)y,$$

or, equivalently,

$$A = -3y^2 + 80y. \tag{21}$$

Note that we have expressed the area A as a function of a single variable y. Also, the function defined by **equation (21)** is quadratic. Compare $A = -3y^2 + 80y$ with the general form $A = ay^2 + by + c$ and note that $a = -3$ and $b = 80$ (we have no need of the fact that $c = 0$). Therefore, if we plot A versus y, the graph is a parabola that opens downward (see **Figure 6**), so the maximum value of A will occur at the vertex. The y-coordinate of the vertex[28] is found with

$$y_{\text{vertex}} = -\frac{b}{2a} = -\frac{80}{2(-3)} = \frac{80}{6} = \frac{40}{3}.$$

To find the width of the rectangular garden, substitute $y = 40/3$ into **equation (20)** and solve for x.

$$x = 80 - 3y = 80 - 3\left(\frac{40}{3}\right) = 80 - 40 = 40. \tag{22}$$

Thus, the width of the rectangular garden is 40 ft. We can find the area of the garden by multiplying the width and the height.

$$A = xy = (40)\left(\frac{40}{3}\right) = \frac{1600}{3} = 533\,\tfrac{1}{3}$$

Note that the resulting area, $A = 533\,\frac{1}{3}\,\text{ft}^2$, is only slightly bigger the last tabular entry found with our numerical experiments.

You can also find the area of the rectangular region by substituting $y = 40/3$ into **equation (21)**. We'll leave it to our readers to check that this provides the same measure for the area. You will also notice that the second coordinate of the vertex in **Figure 6** is the maximum area $A = 1600/3\,\text{ft}^2$.

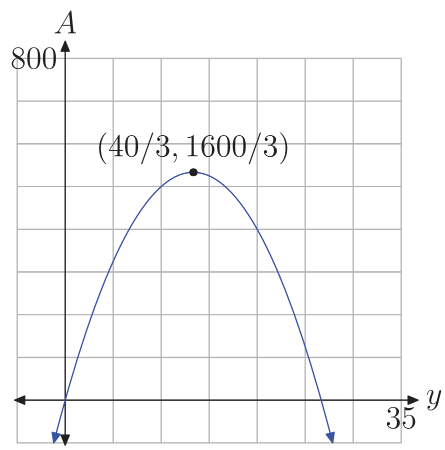


Figure 6. The maximum area, $A = 1600/3\,\text{ft}^2$, occurs at the vertex of the parabola, $(40/3, 1600/3)$.

[28] Because we've plotted A versus y, the horizontal axis is labeled y. Thus, y has taken the usual role of x because the horizontal axis represents the height y of the rectangular garden. That's why we write $y_{\text{vertex}} = -b/(2a)$ instead of $x_{\text{vertex}} = -b/(2a)$ in this example.

2.6 Exercises

1. Find the exact maximum value of the function $f(x) = -x^2 - 3x$.

2. Find the exact maximum value of the function $f(x) = -x^2 - 5x - 2$.

3. Find the vertex of the graph of the function $f(x) = -3x^2 - x - 6$.

4. Find the range of the function $f(x) = -2x^2 - 9x + 2$.

5. Find the exact maximum value of the function $f(x) = -3x^2 - 9x - 4$.

6. Find the equation of the axis of symmetry of the graph of the function $f(x) = -x^2 - 5x - 9$.

7. Find the vertex of the graph of the function $f(x) = 3x^2 + 3x + 9$.

8. Find the exact minimum value of the function $f(x) = x^2 + x + 1$.

9. Find the exact minimum value of the function $f(x) = x^2 + 9x$.

10. Find the range of the function $f(x) = 5x^2 - 3x - 4$.

11. Find the range of the function $f(x) = -3x^2 + 8x - 2$.

12. Find the exact minimum value of the function $f(x) = 2x^2 + 5x - 6$.

13. Find the range of the function $f(x) = 4x^2 + 9x - 8$.

14. Find the exact maximum value of the function $f(x) = -3x^2 - 8x - 1$.

15. Find the equation of the axis of symmetry of the graph of the function $f(x) = -4x^2 - 2x + 9$.

16. Find the exact minimum value of the function $f(x) = 5x^2 + 2x - 3$.

17. A ball is thrown upward at a speed of 8 ft/s from the top of a 182 foot high building. How many seconds does it take for the ball to reach its maximum height? Round your answer to the nearest hundredth of a second.

18. A ball is thrown upward at a speed of 9 ft/s from the top of a 143 foot high building. How many seconds does it take for the ball to reach its maximum height? Round your answer to the nearest hundredth of a second.

19. A ball is thrown upward at a speed of 52 ft/s from the top of a 293 foot high building. What is the maximum height of the ball? Round your answer to the nearest hundredth of a foot.

20. A ball is thrown upward at a speed of 23 ft/s from the top of a 71 foot high building. What is the maximum height of the ball? Round your answer to the nearest hundredth of a foot.

21. Find two numbers whose sum is 20 and whose product is a maximum.

22. Find two numbers whose sum is 36 and whose product is a maximum.

23. Find two numbers whose difference is 12 and whose product is a minimum.

24. Find two numbers whose difference is 24 and whose product is a minimum.

25. One number is 3 larger than twice a second number. Find two such numbers so that their product is a minimum.

26. One number is 2 larger than 5 times a second number. Find two such numbers so that their product is a minimum.

27. Among all pairs of numbers whose sum is -10, find the pair such that the sum of their squares is the smallest possible.

28. Among all pairs of numbers whose sum is -24, find the pair such that the sum of their squares is the smallest possible.

29. Among all pairs of numbers whose sum is 14, find the pair such that the sum of their squares is the smallest possible.

30. Among all pairs of numbers whose sum is 12, find the pair such that the sum of their squares is the smallest possible.

31. Among all rectangles having perimeter 40 feet, find the dimensions (length and width) of the one with the greatest area.

32. Among all rectangles having perimeter 100 feet, find the dimensions (length and width) of the one with the greatest area.

33. A farmer with 1700 meters of fencing wants to enclose a rectangular plot that borders on a river. If no fence is required along the river, what is the largest area that can be enclosed?

34. A rancher with 1500 meters of fencing wants to enclose a rectangular plot that borders on a river. If no fence is required along the river, and the side parallel to the river is x meters long, find the value of x which will give the largest area of the rectangle.

35. A park ranger with 400 meters of fencing wants to enclose a rectangular plot that borders on a river. If no fence is required along the river, and the side parallel to the river is x meters long, find the value of x which will give the largest area of the rectangle.

36. A rancher with 1000 meters of fencing wants to enclose a rectangular plot that borders on a river. If no fence is required along the river, what is the largest area that can be enclosed?

37. Let x represent the demand (the number the public will buy) for an object and let p represent the object's unit price (in dollars). Suppose that the unit price and the demand are linearly related by the equation $p = (-1/3)x + 40$.

a) Express the revenue R (the amount earned by selling the objects) as a function of the demand x.

b) Find the demand that will maximize the revenue.

c) Find the unit price that will maximize the revenue.

d) What is the maximum revenue?

38. Let x represent the demand (the number the public will buy) for an object and let p represent the object's unit price (in dollars). Suppose that the unit price and the demand are linearly related by the equation $p = (-1/5)x + 200$.

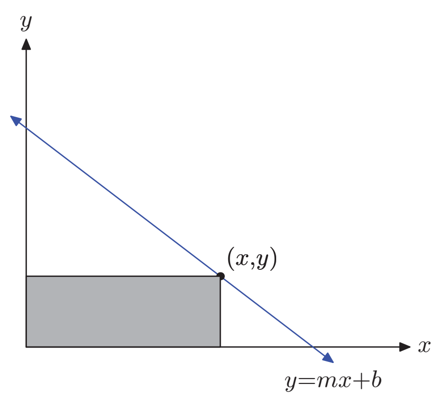
a) Express the revenue R (the amount earned by selling the objects) as a function of the demand x.

b) Find the demand that will maximize the revenue.

c) Find the unit price that will maximize the revenue.

d) What is the maximum revenue?

39. A point from the first quadrant is selected on the line $y = mx + b$. Lines are drawn from this point parallel to the axes to form a rectangle under the line in the first quadrant. Among all such rectangles, find the dimensions of the rectangle with maximum area. What is the maximum area? *Assume* $m < 0$.



40. A rancher wishes to fence a rectangular area. The east-west sides of the rectangle will require stronger support due to prevailing east-west storm winds. Consequently, the cost of fencing for the east-west sides of the rectangular area is \$18 per foot. The cost for fencing the north-south sides of the rectangular area is \$12 per foot. Find the dimension of the largest possible rectangular area that can be fenced for \$7200.

2.6 Answers

1. $\frac{9}{4}$

3. $\left(-\frac{1}{6}, -\frac{71}{12}\right)$

5. $\frac{11}{4}$

7. $\left(-\frac{1}{2}, \frac{33}{4}\right)$

9. $-\frac{81}{4}$

11. $\left(-\infty, \frac{10}{3}\right] = \left\{x \,\middle|\, x \leq \frac{10}{3}\right\}$

13. $\left[-\frac{209}{16}, \infty\right) = \left\{x \,\middle|\, x \geq -\frac{209}{16}\right\}$

15. $x = -\frac{1}{4}$

17. 0.25

19. 335.25

21. 10 and 10

23. 6 and -6

25. $3/2$ and $-3/4$

27. -5, -5

29. 7, 7

31. 10 feet by 10 feet

33. 361250 square meters

35. 200

37.

a) $R = (-1/3)x^2 + 40x$

b) $x = 60$ objects

c) $p = 20$ dollars

d) $R = \$1200$

39. $x = -b/(2m)$, $y = b/2$, $A = -b^2/(4m)$

Chapter 3 Rational Functions

In this chapter, we begin our study of *rational functions* — functions of the form $p(x)/q(x)$, where p and q are both polynomials. Rational functions are similar in structure to rational numbers (commonly thought of as fractions), and they are studied and used extensively in mathematics, engineering, and science.

We will learn how to manipulate these functions, and discover the myriad algebraic tricks and pitfalls that accompany them. We will also see some of the ways that they can be applied to everyday situations, such as modeling the length of time it takes a group of people to complete a task, or calculating the distance traveled by an object.

In more advanced mathematics courses, such as college algebra and calculus, you will learn even more about the intricate nature of rational functions. In many science and engineering courses, you will use rational functions to model what you are studying. In your everyday life, you can use rational functions for a number of useful calculations, such as the amount of time or work that a given task might require. For these reasons, along with the fact that learning how to manipulate rational functions will further your understanding of mathematics, this chapter warrants a good deal of attention.

3.1 Introducing Rational Functions

In a previous chapter, we studied polynomials, functions having equation form

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n. \tag{1}$$

Even though this polynomial is presented in *ascending* powers of x, the leading term of the polynomial is still a_nx^n, the term with the highest power of x. The degree of the polynomial is the highest power of x present, so in this case, the degree of the polynomial is n.

In this section, our study will lead us to the *rational* functions. Note the root word "ratio" in the term "rational." Does it remind you of the word "fraction"? It should, as rational functions are functions in a very specific fractional form.

Definition 2. *A rational function is a function that can be written as a quotient of two polynomial functions. In symbols, the function*

$$f(x) = \frac{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n}{b_0 + b_1x + b_2x^2 + \cdots + b_mx^m} \tag{3}$$

is called a rational function.

For example,

$$f(x) = \frac{1+x}{x+2}, \quad g(x) = \frac{x^2-2x-3}{x+4}, \quad \text{and} \quad h(x) = \frac{3-2x-x^2}{x^3+2x^2-3x-5} \tag{4}$$

are rational functions, while

$$f(x) = \frac{1+\sqrt{x}}{x^2+1}, \quad g(x) = \frac{x^2+2x-3}{1+x^{1/2}-3x^2}, \quad \text{and} \quad h(x) = \sqrt{\frac{x^2-2x-3}{x^2+4x-12}} \tag{5}$$

are **not** rational functions.

Each of the functions in **equation (4)** are rational functions, because in each case, the numerator and denominator of the given expression is a valid polynomial.

However, in **equation (5)**, the numerator of $f(x)$ is not a polynomial (polynomials do not allow the square root of the independent variable). Therefore, f is not a rational function.

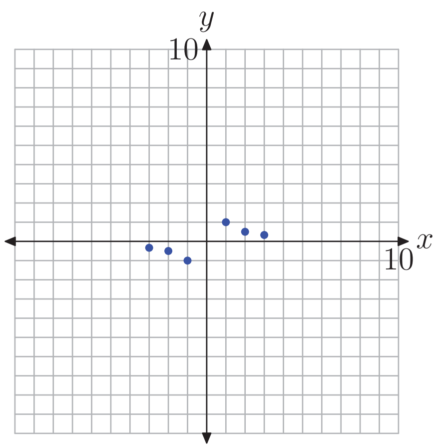
Similarly, the denominator of $g(x)$ in **equation (5)** is not a polynomial. Fractions are not allowed as exponents in polynomials. Thus, g is not a rational function.

Finally, in the case of function h in **equation (5)**, although the radicand (the expression inside the radical) is a rational function, the square root prevents h from being a rational function.

An important skill to develop is the ability to draw the graph of a rational function. Let's begin by drawing the graph of one of the simplest (but most fundamental) rational functions.

The Graph of $y = 1/x$

In all new situations, when we are presented with an equation whose graph we've not considered or do not recognize, we begin the process of drawing the graph by creating a table of points that satisfy the equation. It's important to remember that the graph of an equation is the set of all points that satisfy the equation. We note that zero is not in the domain of $y = 1/x$ (division by zero makes no sense and is not defined), and create a table of points satisfying the equation shown in **Figure 1**.



x	$y = 1/x$
-3	$-1/3$
-2	$-1/2$
-1	-1
1	1
2	$1/2$
3	$1/3$

Figure 1. At the right is a table of points satisfying the equation $y = 1/x$. These points are plotted as solid dots on the graph at the left.

At this point (see **Figure 1**), it's pretty clear what the graph is doing between $x = -3$ and $x = -1$. Likewise, it's clear what is happening between $x = 1$ and $x = 3$. However, there are some open areas of concern.

1. What happens to the graph as x increases without bound? That is, what happens to the graph as x moves toward ∞?
2. What happens to the graph as x decreases without bound? That is, what happens to the graph as x moves toward $-\infty$?
3. What happens to the graph as x approaches zero from the right?
4. What happens to the graph as x approaches zero from the left?

Let's answer each of these questions in turn. We'll begin by discussing the "end-behavior" of the rational function defined by $y = 1/x$. First, the right end. What happens as x increases without bound? That is, what happens as x increases toward ∞? In **Table 1**(a), we computed $y = 1/x$ for x equalling 100, 1 000, and 10 000. Note how the y-values in **Table 1**(a) are all positive and approach zero.

Students in calculus use the following notation for this idea.

$$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{1}{x} = 0 \tag{6}$$

They say "the limit of y as x approaches infinity is zero." That is, as x approaches infinity, y approaches zero.

x	$y = 1/x$
100	0.01
1 000	0.001
10 000	0.0001

(a)

x	$y = 1/x$
−100	−0.01
−1 000	−0.001
−10 000	−0.0001

(b)

Table 1. Examining the end-behavior of $y = 1/x$.

A completely similar event happens at the left end. As x decreases without bound, that is, as x decreases toward $-\infty$, note that the y-values in **Table 1**(b) are all negative and approach zero. Calculus students have a similar notation for this idea.

$$\lim_{x\to-\infty} y = \lim_{x\to-\infty} \frac{1}{x} = 0. \tag{7}$$

They say "the limit of y as x approaches negative infinity is zero." That is, as x approaches negative infinity, y approaches zero.

These numbers in **Tables 1**(a) and **1**(b), and the ideas described above, predict the correct end-behavior of the graph of $y = 1/x$. At each end of the x-axis, the y-values must approach zero. This means that the graph of $y = 1/x$ must approach the x-axis for x-values at the far right- and left-ends of the graph. In this case, we say that the x-axis acts as a *horizontal asymptote* for the graph of $y = 1/x$. As x approaches either positive or negative infinity, the graph of $y = 1/x$ approaches the x-axis. This behavior is shown in **Figure 2**.

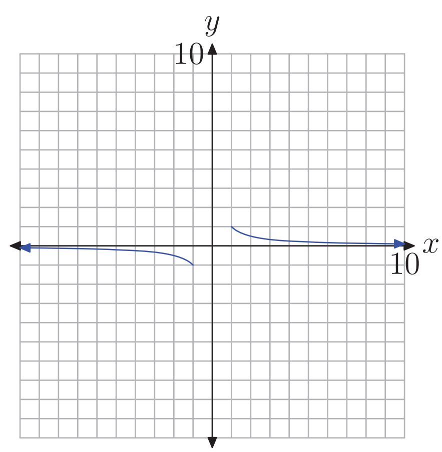
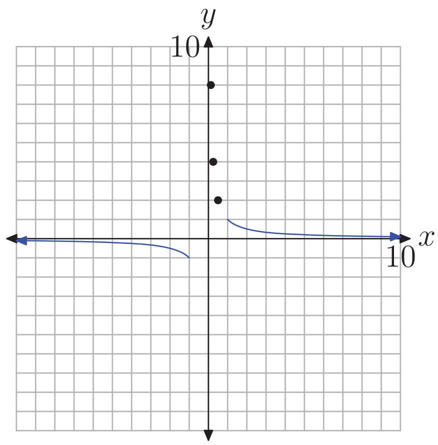


Figure 2. The graph of $1/x$ approaches the x-axis as x increases or decreases without bound.

Our last investigation will be on the interval from $x = -1$ to $x = 1$. Readers are again reminded that the function $y = 1/x$ is undefined at $x = 0$. Consequently, we will break this region in half, first investigating what happens on the region between $x = 0$

and $x = 1$. We evaluate $y = 1/x$ at $x = 1/2$, $x = 1/4$, and $x = 1/8$, as shown in the table in **Figure 3**, then plot the resulting points.



x	$y = 1/x$
$1/2$	2
$1/4$	4
$1/8$	8

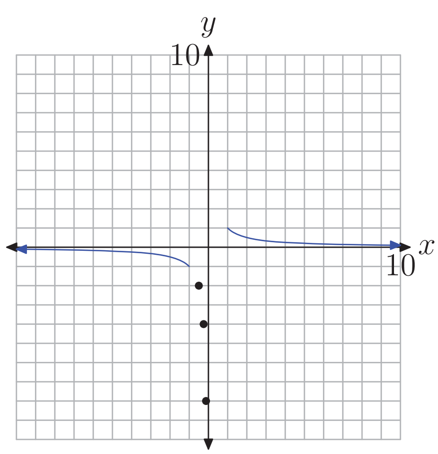
Figure 3. At the right is a table of points satisfying the equation $y = 1/x$. These points are plotted as solid dots on the graph at the left.

Note that the x-values in the table in **Figure 3** approach zero from the right, then note that the corresponding y-values are getting larger and larger. We could continue in this vein, adding points. For example, if $x = 1/16$, then $y = 16$. If $x = 1/32$, then $y = 32$. If $x = 1/64$, then $y = 64$. Each time we halve our value of x, the resulting value of x is closer to zero, and the corresponding y-value doubles in size. Calculus students describe this behavior with the notation

$$\lim_{x \to 0^+} y = \lim_{x \to 0^+} \frac{1}{x} = \infty. \tag{8}$$

That is, as "x approaches zero from the right, the value of y grows to infinity." This is evident in the graph in **Figure 3**, where we see the plotted points move closer to the vertical axis while at the same time moving upward without bound.

A similar thing happens on the other side of the vertical axis, as shown in **Figure 4**.



x	$y = 1/x$
$-1/2$	-2
$-1/4$	-4
$-1/8$	-8

Figure 4. At the right is a table of points satisfying the equation $y = 1/x$. These points are plotted as solid dots on the graph at the left.

Again, calculus students would write

$$\lim_{x\to 0^-} y = \lim_{x\to 0^-} \frac{1}{x} = -\infty. \tag{9}$$

That is, "as x approaches zero from the left, the values of y decrease to negative infinity." In **Figure 4**, it is clear that as points move closer to the vertical axis (as x approaches zero) from the left, the graph decreases without bound.

The evidence gathered to this point indicates that the vertical axis is acting as a *vertical asymptote*. As x approaches zero from either side, the graph approaches the vertical axis, either rising to infinity, or falling to negative infinity. The graph cannot cross the vertical axis because the function is undefined there. The completed graph is shown in **Figure 5**.

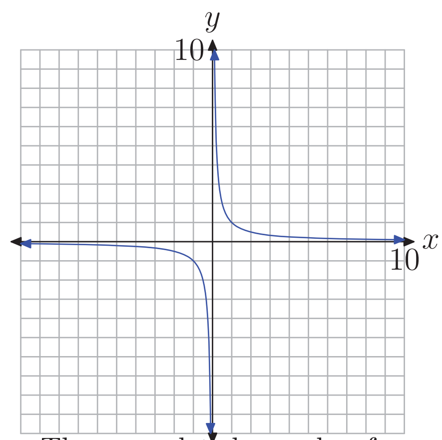


Figure 5. The completed graph of $y = 1/x$. Note how the x-axis acts as a horizontal asymptote, while the y-axis acts as a vertical asymptote.

The complete graph of $y = 1/x$ in **Figure 5** is called a *hyperbola* and serves as a fundamental starting point for all subsequent discussion in this section.

We noted earlier that the domain of the function defined by the equation $y = 1/x$ is the set $D = \{x : x \neq 0\}$. Zero is excluded from the domain because division by zero is undefined. It's no coincidence that the graph has a vertical asymptote at $x = 0$. We'll see this relationship reinforced in further examples.

Translations

In this section, we will translate the graph of $y = 1/x$ in both the horizontal and vertical directions.

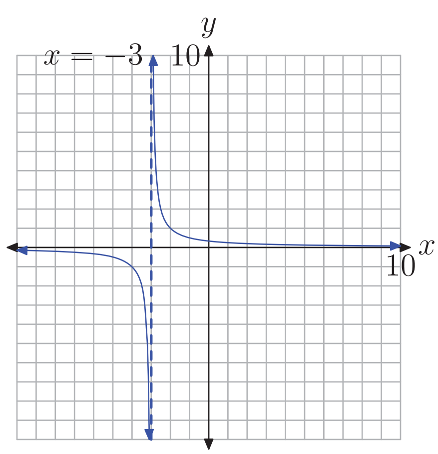
I **Example 10.** *Sketch the graph of*

$$y = \frac{1}{x+3} - 4. \tag{11}$$

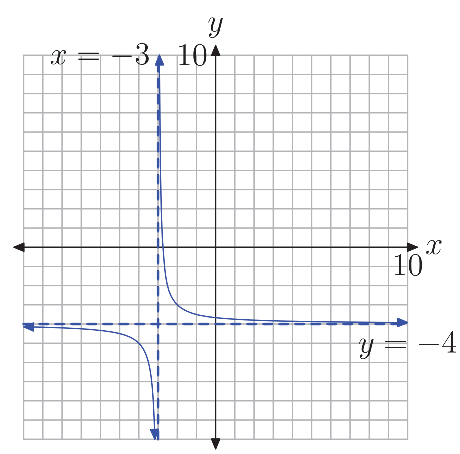
Technically, the function defined by $y = 1/(x+3) - 4$ does not have the general form (**3**) of a rational function. However, in later chapters we will show how $y = 1/(x+3) - 4$ can be manipulated into the general form of a rational function.

We know what the graph of $y = 1/x$ looks like. If we replace x with $x + 3$, this will shift the graph of $y = 1/x$ three units to the left, as shown in **Figure 6**(a). Note that the vertical asymptote has also shifted 3 units to the left of its original position (the y-axis) and now has equation $x = -3$. By tradition, we draw the vertical asymptote as a dashed line.

If we subtract 4 from the result in **Figure 6**(a), this will shift the graph in **Figure 6**(a) four units downward to produce the graph shown in **Figure 6**(b). Note that the horizontal asymptote also shifted 4 units downward from its original position (the x-axis) and now has equation $y = -4$.



(a) $y = 1/(x+3)$



(b) $y = 1/(x+3) - 4$

Figure 6. Shifting the graph of $y = 1/x$.

If you examine **equation (11)**, you note that you cannot use $x = -3$ as this will make the denominator of **equation (11)** equal to zero. In **Figure 6**(b), note that there is a vertical asymptote in the graph of **equation (11)** at $x = -3$. This is a common occurrence, which will be a central theme of this chapter.

Let's ask another key question.

I **Example 12.** *What are the domain and range of the rational function presented in* **Example 10***?*

You can glance at the equation

$$y = \frac{1}{x+3} - 4$$

of **Example 10** and note that $x = -3$ makes the denominator zero and must be excluded from the domain. Hence, the domain of this function is $D = \{x : x \neq -3\}$.

However, you can also determine the domain by examining the graph of the function in **Figure 6**(b). Note that the graph extends indefinitely to the left and right. One might first guess that the domain is all real numbers if it were not for the vertical asymptote at $x = -3$ interrupting the continuity of the graph. Because the graph of the function gets arbitrarily close to this vertical asymptote (on either side) without actually touching the asymptote, the graph does not contain a point having an x-value equaling -3. Hence, the domain is as above, $D = \{x : x \neq -3\}$. This is comforting that the graphical analysis agrees with our earlier analytical determination of the domain.

The graph is especially helpful in determining the range of the function. Note that the graph rises to positive infinity and falls to negative infinity. One would first guess that the range is all real numbers if it were not for the horizontal asymptote at $y = -4$ interrupting the continuity of the graph. Because the graph gets arbitrarily close to the horizontal asymptote (on either side) without actually touching the asymptote, the graph does not contain a point having a y-value equaling -4. Hence, -4 is excluded from the range. That is, $R = \{y : y \neq -4\}$.

Scaling and Reflection

In this section, we will both scale and reflect the graph of $y = 1/x$. For extra measure, we also throw in translations in the horizontal and vertical directions.

I **Example 13.** *Sketch the graph of*

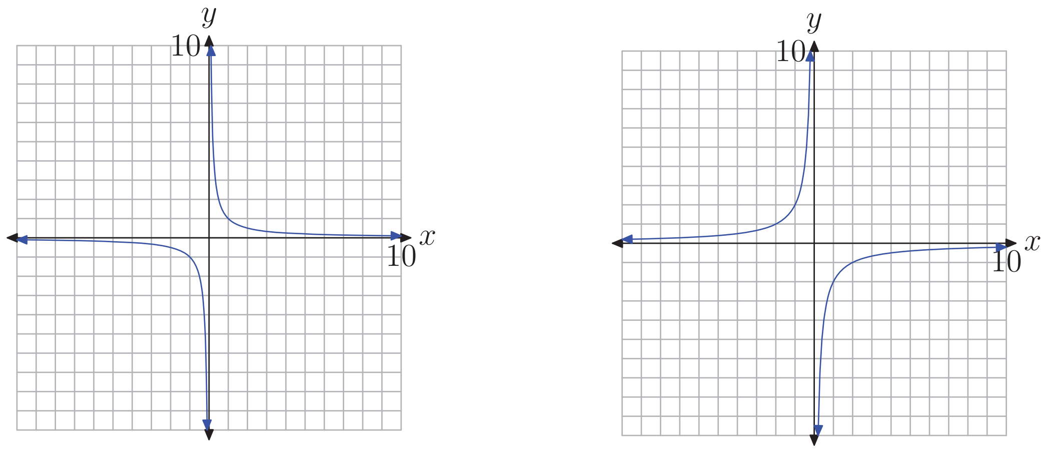
$$y = -\frac{2}{x-4} + 3. \qquad (14)$$

First, we multiply the equation $y = 1/x$ by -2 to get

$$y = -\frac{2}{x}.$$

Multiplying by 2 should stretch the graph in the vertical directions (both positive and negative) by a factor of 2. Note that points that are very near the x-axis, when doubled, are not going to stray too far from the x-axis, so the horizontal asymptote will remain the same. Finally, multiplying by -2 will not only stretch the graph, it will also reflect the graph across the x-axis, as shown in **Figure 7**(b).[2]

[2] Recall that we saw similar behavior when studying the parabola. The graph of $y = -2x^2$ stretched (vertically) the graph of the equation $y = x^2$ by a factor of 2, then reflected the result across the x-axis.



(a) $y = 1/x$ (b) $y = -2/x$

Figure 7. Scaling and reflecting the graph of $y = 1/x$.

Replacing x with $x - 4$ will shift the graph 4 units to the right, then adding 3 will shift the graph 3 units up, as shown in **Figure 8**. Note again that $x = 4$ makes the denominator of $y = -2/(x - 4) + 3$ equal to zero and there is a vertical asymptote at $x = 4$. The domain of this function is $D = \{x : x \neq 4\}$.

As x approaches positive or negative infinity, points on the graph of $y = -2/(x-4)+3$ get arbitrarily close to the horizontal asymptote $y = 3$ but never touch it. Therefore, there is no point on the graph that has a y-value of 3. Thus, the range of the function is the set $R = \{y : y \neq 3\}$.

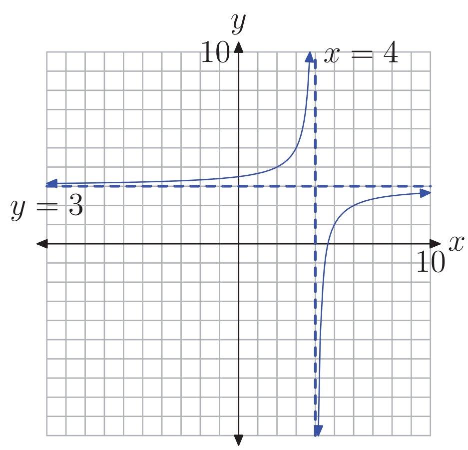


Figure 8. The graph of $y = -2/(x - 4) + 3$ is shifted 4 units right and 3 units up.

Difficulties with the Graphing Calculator

The graphing calculator does a very good job drawing the graphs of "continuous functions."

> A continuous function is one that can be drawn in one continuous stroke, never lifting pen or pencil from the paper during the drawing.

Polynomials, such as the one in **Figure 9**, are continuous functions.

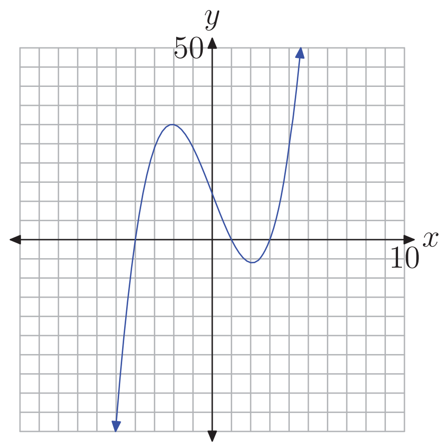


Figure 9. A polynomial is a continuous function.

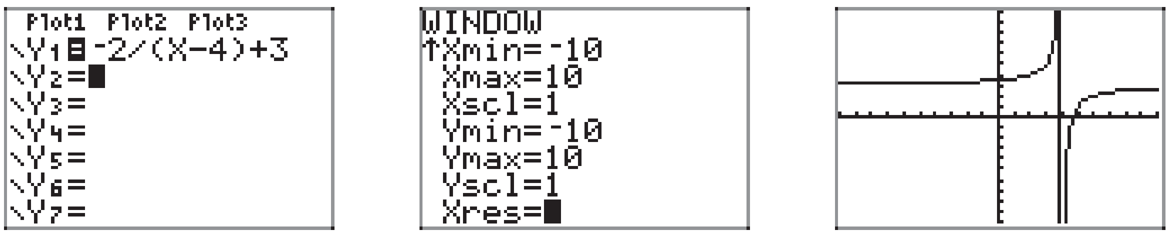
Unfortunately, a rational function with vertical asymptote(s) is not a continuous function. First, you have to lift your pen at points where the denominator is zero, because the function is undefined at these points. Secondly, it's not uncommon to have to jump from positive infinity to negative infinity (or vice-versa) when crossing a vertical asymptote. When this happens, we have to lift our pen and shift it before continuing with our drawing.

However, the graphing calculator does not know how to do this "lifting" of the pen near vertical asymptotes. The graphing calculator only knows one technique, plot a point, then connect it with a segment to the last point plotted, move an incremental distance and repeat. Consequently, when the graphing calculator crosses a vertical asymptote where there is a shift from one type of infinity to another (e.g., from positive to negative), the calculator draws a "false line" of connection, one that it should not draw. Let's demonstrate this aberration with an example.

I **Example 15.** *Use a graphing calculator to draw the graph of the rational function in* **Example 13**.

Load the equation into your calculator, as shown in **Figure 10**(a). Set the window as shown in **Figure 10**(b), then push the `GRAPH` button to draw the graph shown in **Figure 10**(c). Results may differ on some calculators, but in our case, note the "false line" drawn from the top of the screen to the bottom, attempting to "connect" the two branches of the hyperbola.

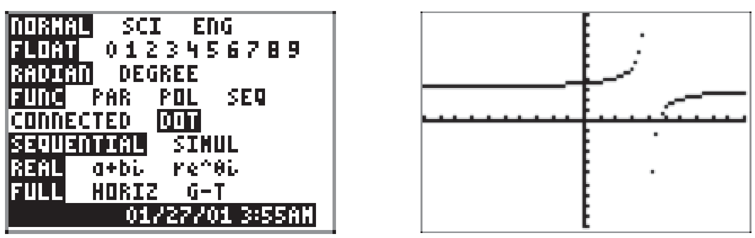
Some might rejoice and claim, "Hey, my graphing calculator draws vertical asymptotes." However, before you get too excited, note that in **Figure 8** the vertical asymptote should occur at *exactly* $x = 4$. If you look very carefully at the "vertical line" in **Figure 10**(c), you'll note that it just misses the tick mark at $x = 4$. This "vertical line" is a line that the calculator should not draw. The calculator is attempting to draw a continuous function where one doesn't exist.



(a) (b) (c)

Figure 10. The calculator attempts to draw a continuous function when it shouldn't.

One possible workaround[3] is to press the `MODE` button on your keyboard, which opens the menu shown in **Figure 11**(a). Use the arrow keys to highlight `DOT` instead of `CONNECTED` and press the `ENTER` key to make the selection permanent. Press the `GRAPH` button to draw the graph in **Figure 11**(b).



(a) (b)

Figure 11. The same graph in "dot mode."

This "dot mode" on your calculator calculates the next point on the graph and plots the point, but it does not connect it with a line segment to the previously plotted point. This mode is useful in demonstrating that the vertical line in **Figure 10**(c) is not really part of the graph, but we lose some parts of the graph we'd really like to see. Compromise is in order.

This example clearly shows that intelligent use of the calculator is a required component of this course. The calculator is not simply a "black box" that automatically does what you want it to do. In particular, when you are drawing rational functions, it helps to know ahead of time the placement of the vertical asymptotes. Knowledge

[3] Instructors might discuss a number of alternative strategies to represent rational functions on the graphing calculator. What we present here is only one of a number of approaches.

of the asymptotes, coupled with what you see on your calculator screen, should enable you to draw a graph as accurate as that shown in **Figure 8**.

Gentle reminder. You'll want to set your calculator back in "connected mode." To do this, press the MODE button on your keyboard to open the menu in **Figure 10**(a) once again. Use your arrow keys to highlight CONNECTED, then press the ENTER key to make the selection permanent.

3.1 Exercises

In **Exercises 1-14**, perform each of the following tasks for the given rational function.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis.
ii. Use geometric transformations as in Examples 10, 12, and 13 to draw the graphs of each of the following rational functions. Draw the vertical and horizontal asymptotes as dashed lines and label each with its equation. You may use your calculator to **check** your solution, but you should be able to draw the rational function without the use of a calculator.
iii. Use set-builder notation to describe the domain and range of the given rational function.

1. $f(x) = -2/x$

2. $f(x) = 3/x$

3. $f(x) = 1/(x-4)$

4. $f(x) = 1/(x+3)$

5. $f(x) = 2/(x-5)$

6. $f(x) = -3/(x+6)$

7. $f(x) = 1/x - 2$

8. $f(x) = -1/x + 4$

9. $f(x) = -2/x - 5$

10. $f(x) = 3/x - 5$

11. $f(x) = 1/(x-2) - 3$

12. $f(x) = -1/(x+1) + 5$

13. $f(x) = -2/(x-3) - 4$

14. $f(x) = 3/(x+5) - 2$

In **Exercises 15-22**, find all vertical asymptotes, if any, of the graph of the given function.

15. $f(x) = -\dfrac{5}{x+1} - 3$

16. $f(x) = \dfrac{6}{x+8} + 2$

17. $f(x) = -\dfrac{9}{x+2} - 6$

18. $f(x) = -\dfrac{8}{x-4} - 5$

19. $f(x) = \dfrac{2}{x+5} + 1$

20. $f(x) = -\dfrac{3}{x+9} + 2$

21. $f(x) = \dfrac{7}{x+8} - 9$

22. $f(x) = \dfrac{6}{x-5} - 8$

In **Exercises 23-30**, find all horizontal asymptotes, if any, of the graph of the given function.

23. $f(x) = \dfrac{5}{x+7} + 9$

24. $f(x) = -\dfrac{8}{x+7} - 4$

25. $f(x) = \frac{8}{x+5} - 1$

26. $f(x) = -\frac{2}{x+3} + 8$

27. $f(x) = \frac{7}{x+1} - 9$

28. $f(x) = -\frac{2}{x-1} + 5$

29. $f(x) = \frac{5}{x+2} - 4$

30. $f(x) = -\frac{6}{x-1} - 2$

In **Exercises 31-38**, state the domain of the given rational function using set-builder notation.

31. $f(x) = \frac{4}{x+5} + 5$

32. $f(x) = -\frac{7}{x-6} + 1$

33. $f(x) = \frac{6}{x-5} + 1$

34. $f(x) = -\frac{5}{x-3} - 9$

35. $f(x) = \frac{1}{x+7} + 2$

36. $f(x) = -\frac{2}{x-5} + 4$

37. $f(x) = -\frac{4}{x+2} + 2$

38. $f(x) = \frac{2}{x+6} + 9$

In **Exercises 39-46**, find the range of the given function, and express your answer in set notation.

39. $f(x) = \frac{2}{x-3} + 8$

40. $f(x) = \frac{4}{x-3} + 5$

41. $f(x) = -\frac{5}{x-8} - 5$

42. $f(x) = -\frac{2}{x+1} + 6$

43. $f(x) = \frac{7}{x+7} + 5$

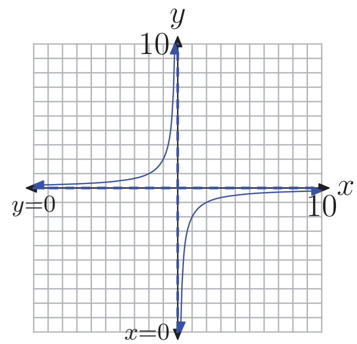
44. $f(x) = -\frac{8}{x+3} + 9$

45. $f(x) = \frac{4}{x+3} - 2$

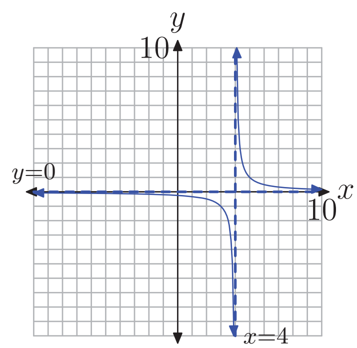
46. $f(x) = -\frac{5}{x-4} + 9$

3.1 Answers

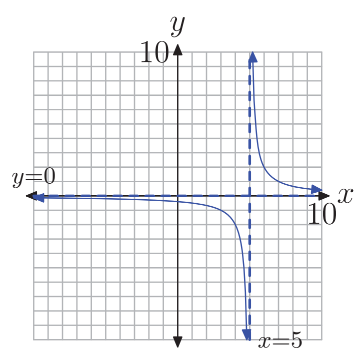
1. $D = \{x : x \neq 0\}$, $R = \{y : y \neq 0\}$



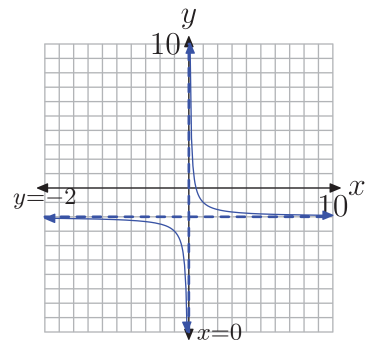
3. $D = \{x : x \neq 4\}$, $R = \{y : y \neq 0\}$



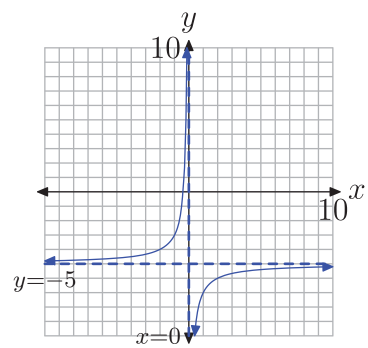
5. $D = \{x : x \neq 5\}$, $R = \{y : y \neq 0\}$



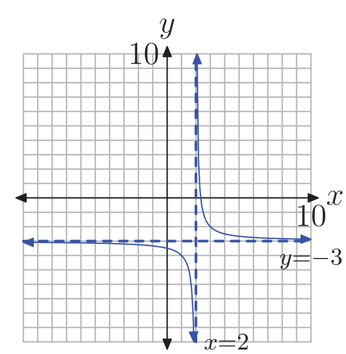
7. $D = \{x : x \neq 0\}$, $R = \{y : y \neq -2\}$



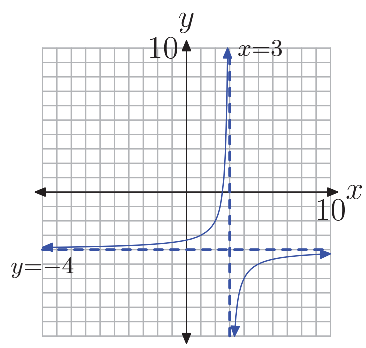
9. $D = \{x : x \neq 0\}$, $R = \{y : y \neq -5\}$



11. $D = \{x : x \neq 2\}$, $R = \{y : y \neq -3\}$



13. $D = \{x : x \neq 3\}$, $R = \{y : y \neq -4\}$



15. Vertical asymptote: $x = -1$

17. Vertical asymptote: $x = -2$

19. Vertical asymptote: $x = -5$

21. Vertical asymptote: $x = -8$

23. Horizontal asymptote: $y = 9$

25. Horizontal asymptote: $y = -1$

27. Horizontal asymptote: $y = -9$

29. Horizontal asymptote: $y = -4$

31. Domain $= \{x : x \neq -5\}$

33. Domain $= \{x : x \neq 5\}$

35. Domain $= \{x : x \neq -7\}$

37. Domain $= \{x : x \neq -2\}$

39. Range $= \{y : y \neq 8\}$

41. Range $= \{y : y \neq -5\}$

43. Range $= \{y : y \neq 5\}$

45. Range $= \{y : y \neq -2\}$

3.2 Reducing Rational Functions

The goal of this section is to learn how to reduce a rational expression to "lowest terms." Of course, that means that we will have to understand what is meant by the phrase "lowest terms." With that thought in mind, we begin with a discussion of the *greatest common divisor* of a pair of integers.

First, we define what we mean by "divisibility."

Definition 1. *Suppose that we have a pair of integers a and b. We say that "a is a divisor of b," or "a divides b" if and only if there is another integer k so that $b = ak$. Another way of saying the same thing is to say that a divides b if, upon dividing b by a, the remainder is zero.*

Let's look at an example.

I **Example 2.** *What are the divisors of 12?*

Because $12 = 1 \times 12$, both 1 and 12 are divisors[6] of 12. Because $12 = 2 \times 6$, both 2 and 6 are divisors of 12. Finally, because $12 = 3 \times 4$, both 3 and 4 are divisors of 12. If we list them in ascending order, the divisors of 12 are

$$1,\ 2,\ 3,\ 4,\ 6,\ \text{and } 12.$$

Let's look at another example.

I **Example 3.** *What are the divisors of 18?*

Because $18 = 1 \times 18$, both 1 and 18 are divisors of 18. Similarly, $18 = 2 \times 9$ and $18 = 3 \times 6$, so in ascending order, the divisors of 18 are

$$1,\ 2,\ 3,\ 6,\ 9,\ \text{and } 18.$$

The *greatest common divisor* of two or more integers is the largest divisor the integers share in common. An example should make this clear.

I **Example 4.** *What is the greatest common divisor of 12 and 18?*

In **Example 2** and **Example 3**, we saw the following.

$$\begin{aligned}\text{Divisors of } 12 &: \boxed{1},\ \boxed{2},\ \boxed{3},\ 4,\ \boxed{6},\ 12\\ \text{Divisors of } 18 &: \boxed{1},\ \boxed{2},\ \boxed{3},\ \boxed{6},\ 9,\ 18\end{aligned}$$

[6] The word "divisor" and the word "factor" are synonymous.

We've framed the divisors that 12 and 18 have in common. They are 1, 2, 3, and 6. The "greatest" of these "common" divisors is 6. Hence, we say that "the greatest common divisor of 12 and 18 is 6."

Definition 5. *The greatest common divisor of two integers a and b is the largest divisor they have in common. We will use the notation*

$$GCD(a, b)$$

to represent the greatest common divisor of a and b.

Thus, as we saw in **Example 4**, $\text{GCD}(12, 18) = 6$.

When the greatest common divisor of a pair of integers is one, we give that pair a special name.

Definition 6. *Let a and b be integers. If the greatest common divisor of a and b is one, that is, if $GCD(a, b) = 1$, then we say that a and b are* **relatively prime**.

For example:

- 9 and 12 are **not** relatively prime because $\text{GCD}(9, 12) = 3$.
- 10 and 15 are **not** relatively prime because $\text{GCD}(10, 15) = 5$.
- 8 and 21 **are** relatively prime because $\text{GCD}(8, 21) = 1$.

We can now define what is meant when we say that a rational number is reduced to lowest terms.

Definition 7. *A rational number in the form p/q, where p and q are integers, is said to be reduced to lowest terms if and only if $GCD(p, q) = 1$. That is, p/q is reduced to lowest terms if the greatest common divisor of both numerator and denominator is 1.*

As we saw in **Example 4**, the greatest common divisor of 12 and 18 is 6. Therefore, the fraction 12/18 is **not** reduced to lowest terms. However, we can reduce 12/18 to lowest terms by dividing both numerator and denominator by their greatest common divisor. That is,

$$\frac{12}{18} = \frac{12 \div 6}{18 \div 6} = \frac{2}{3}.$$

Note that $\text{GCD}(2, 3) = 1$, so 2/3 is reduced to lowest terms.

When it is difficult to ascertain the greatest common divisor, we'll find it more efficient to proceed as follows:

- Prime factor both numerator and denominator.
- Cancel common factors.

Thus, to reduce 12/18 to lowest terms, first express both numerator and denominator as a product of prime numbers, then cancel common primes.

$$\frac{12}{18} = \frac{2 \cdot 2 \cdot 3}{2 \cdot 3 \cdot 3} = \frac{\cancel{2} \cdot 2 \cdot \cancel{3}}{\cancel{2} \cdot 3 \cdot \cancel{3}} = \frac{2}{3} \tag{8}$$

When you cancel a 2, you're actually dividing both numerator and denominator by 2. When you cancel a 3, you're actually dividing both numerator and denominator by 3. Note that doing both (dividing by 2 and then dividing by 3) is equivalent to dividing both numerator and denominator by 6.

We will favor this latter technique, precisely because it is identical to the technique we will use to reduce rational functions to lowest terms. However, this "cancellation" technique has some pitfalls, so let's take a moment to discuss some common cancellation mistakes.

Cancellation

You can spark some pretty heated debate amongst mathematics educators by innocently mentioning the word "cancellation." There seem to be two diametrically opposed camps, those who don't mind when their students use the technique of cancellation, and on the other side, those that refuse to even use the term "cancellation" in their classes.

Both sides of the argument have merit. As we showed in **equation (8)**, we can reduce 12/18 quite efficiently by simply canceling common factors. On the other hand, instructors from the second camp prefer to use the phrase "factor out a 1" instead of the phrase "cancel," encouraging their students to reduce 12/18 as follows.

$$\frac{12}{18} = \frac{2 \cdot 2 \cdot 3}{2 \cdot 3 \cdot 3} = \frac{2}{3} \cdot \frac{2 \cdot 3}{2 \cdot 3} = \frac{2}{3} \cdot 1 = \frac{2}{3}$$

This is a perfectly valid technique and one that, quite honestly, avoids the quicksand of "cancellation mistakes." Instructors who grow weary of watching their students "cancel" when they shouldn't are quite likely to promote this latter technique.

However, if we can help our students avoid "cancellation mistakes," we prefer to allow our students to cancel common factors (as we did in **equation (8)**) when reducing fractions such as 12/18 to lowest terms. So, with these thoughts in mind, let's discuss some of the most common cancellation mistakes.

Let's begin with a most important piece of advice.

How to Avoid Cancellation Mistakes. You may only cancel factors, not addends. To avoid cancellation mistakes, factor **completely** before you begin to cancel.

Warning 9. *Many of the ensuing calculations are incorrect. They are examples of common mistakes that are made when performing cancellation. Make sure that you read carefully and avoid just "scanning" these calculations.*

As a first example, consider the rational expression

$$\frac{2+6}{2},$$

which clearly equals 8/2, or 4. However, if you cancel in this situation, as in

$$\frac{2+6}{2} = \frac{\cancel{2}+6}{\cancel{2}}, \tag{10}$$

you certainly do not get the same result. So, what happened?

Note that in the numerator of **equation (10)**, the 2 and the 6 are separated by a plus sign. Thus, they are not factors; they are addends! You are not allowed to cancel addends, only factors.

Suppose, for comparison, that the rational expression had been

$$\frac{2\cdot 6}{2},$$

which clearly equals 12/2, or 6. In this case, the 2 and the 6 in the numerator are separated by a multiplication symbol, so they are factors and cancellation is allowed, as in

$$\frac{2\cdot 6}{2} = \frac{\cancel{2}\cdot 6}{\cancel{2}} = 6. \tag{11}$$

Now, before you dismiss these examples as trivial, consider the following examples which are identical in structure. First, consider

$$\frac{x+(x+2)}{x} = \frac{\cancel{x}+(x+2)}{\cancel{x}} = x+2.$$

This cancellation is identical to that performed in **equation (10)** and is not allowed. In the numerator, note that x and $(x+2)$ are separated by an addition symbol, so they are addends. You are not allowed to cancel addends!

Conversely, consider the following example.

$$\frac{x(x+2)}{x} = \frac{\cancel{x}(x+2)}{\cancel{x}} = x+2$$

In the numerator of this example, x and $(x+2)$ are separated by implied multiplication. Hence, they are factors and cancellation is permissible.

Look again at **equation (10)**, where the correct answer should have been 8/2, or 4. We mistakenly found the answer to be 6, because we cancelled addends. A workaround would be to first factor the numerator of **equation (10)**, then cancel, as follows.

$$\frac{2+6}{2} = \frac{2(1+3)}{2} = \frac{\cancel{2}(1+3)}{\cancel{2}} = 1+3 = 4$$

Note that we cancelled **factors** in this approach, which is permissible, and got the correct answer 4.

Warning 12. *We are finished discussing common cancellation mistakes and you may not continue reading with confidence that all mathematics is correctly presented.*

Reducing Rational Expressions in x

Now that we've discussed some fundamental ideas and techniques, let's apply what we've learned to rational expressions that are functions of an independent variable (usually x). Let's start with a simple example.

► **Example 13.** *Reduce the rational expression*

$$\frac{2x-6}{x^2-7x+12} \tag{14}$$

to lowest terms. For what values of x is your result valid?

In the numerator, factor out a 2, as in $2x - 6 = 2(x - 3)$.

The denominator is a quadratic trinomial with $ac = (1)(12) = 12$. The integer pair -3 and -4 has product 12 and sum -7, so the denominator factors as shown.

$$\frac{2x-6}{x^2-7x+12} = \frac{2(x-3)}{(x-3)(x-4)}.$$

Now that both numerator and denominator are factored, we can cancel common factors.

$$\frac{2x-6}{x^2-7x+12} = \frac{2\cancel{(x-3)}}{\cancel{(x-3)}(x-4)} = \frac{2}{x-4}$$

Thus, we have shown that

$$\frac{2x-6}{x^2-7x+12} = \frac{2}{x-4}. \tag{15}$$

In **equation (15)**, we are stating that the expression on the left (the original expression) is *identical* to the expression on the right for all values of x.

Actually, there are two notable exceptions, the first of which is $x = 3$. If we substitute $x = 3$ into the left-hand side of **equation (15)**, we get

$$\frac{2x-6}{x^2-7x+12} = \frac{2(3)-6}{(3)^2-7(3)+12} = \frac{0}{0}$$

We cannot divide by zero, so the left-hand side of **equation (15)** is undefined if $x = 3$. Therefore, the result in **equation (15)** is not valid if $x = 3$.

Similarly, if we insert $x = 4$ in the left-hand side of **equation (15)**,

$$\frac{2x-6}{x^2-7x+12} = \frac{2(4)-6}{(4)^2-7(4)+12} = \frac{2}{0}.$$

Again, division by zero is undefined. The left-hand side of **equation (15)** is undefined if $x = 4$, so the result in **equation (15)** is not valid if $x = 4$. Note that the right-hand side of **equation (15)** is also undefined at $x = 4$.

However, the algebraic work we did above guarantees that the left-hand side of **equation (15)** will be identical to the right-hand side of **equation (15)** for all other values of x. For example, if we substitute $x = 5$ into the left-hand side of **equation (15)**,

$$\frac{2x-6}{x^2-7x+12} = \frac{2(5)-6}{(5)^2-7(5)+12} = \frac{4}{2} = 2.$$

On the other hand, if we substitute $x = 5$ into the right-hand side of **equation (15)**,

$$\frac{2}{x-4} = \frac{2}{5-4} = 2.$$

Hence, both sides of **equation (15)** are identical when $x = 5$. In a similar manner, we could check the validity of the identity in **equation (15)** for all other values of x.

You can use the graphing calculator to verify the identity in **equation (15)**. Load the left- and right-hand sides of **equation (15)** in `Y=` menu, as shown in **Figure 1**(a). Press `2nd TBLSET` and adjust settings as shown in **Figure 1**(b). Be sure that you highlight `AUTO` for both independent and dependent variables and press `ENTER` on each to make the selection permanent. In **Figure 1**(b), note that we've set `TblStart = 0` and Δ`Tbl` $=$ `1`. Press `2nd TABLE` to produce the tabular results shown in **Figure 1**(c).

(a) (b) (c)

Figure 1. Using the graphing calculator to check that the left- and right-hand sides of **equation (15)** are identical.

Remember that we placed the left- and right-hand sides of **equation (15)** in `Y1` and `Y2`, respectively.

- In the tabular results of **Figure 1**(c), note the `ERR` (error) message in `Y1` when $x = 3$ and $x = 4$. This agrees with our findings above, where the left-hand side of **equation (15)** was undefined because of the presence of zero in the denominator when $x = 3$ or $x = 4$.
- In the tabular results of **Figure 1**(c), note that the value of `Y1` and `Y2` agree for all other values of x.

We are led to the following key result.

Restrictions. In general, when you reduce a rational expression to lowest terms, the expression obtained should be **identical** to the original expression for all values of the variables in each expression, save those values of the variables that make any denominator equal to zero. This applies to the denominator in the original expression, all intermediate expressions in your work, and the final result. We will refer to any values of the variable that make any denominator equal to zero as **restrictions**.

Let's look at another example.

► **Example 16.** *Reduce the expression*

$$\frac{2x^2+5x-12}{4x^3+16x^2-9x-36} \tag{17}$$

to lowest terms. State all restrictions.

The numerator is a quadratic trinomial with $ac = (2)(-12) = -24$. The integer pair -3 and 8 have product -24 and sum 5. Break the middle term of the polynomial in the numerator into a sum using this integer pair, then factor by grouping.

$$\begin{aligned} 2x^2+5x-12 &= 2x^2-3x+8x-12 \\ &= x(2x-3)+4(2x-3) \\ &= (x+4)(2x-3) \end{aligned}$$

Factor the denominator by grouping.

$$\begin{aligned} 4x^3+16x^2-9x-36 &= 4x^2(x+4)-9(x+4) \\ &= (4x^2-9)(x+4) \\ &= (2x+3)(2x-3)(x+4) \end{aligned}$$

Note how the difference of two squares pattern was used to factor $4x^2-9 = (2x+3)(2x-3)$ in the last step.

Now that we've factored both numerator and denominator, we cancel common factors.

$$\begin{aligned} \frac{2x^2+5x-12}{4x^3+16x^2-9x-36} &= \frac{(x+4)(2x-3)}{(2x+3)(2x-3)(x+4)} \\ &= \frac{\cancel{(x+4)}\cancel{(2x-3)}}{(2x+3)\cancel{(2x-3)}\cancel{(x+4)}} \\ &= \frac{1}{2x+3} \end{aligned}$$

We must now determine the restrictions. This means that we must find those values of x that make any denominator equal to zero.

- In the body of our work, we have the denominator $(2x+3)(2x-3)(x+4)$. If we set this equal to zero, the zero product property implies that

$$2x+3=0 \quad \text{or} \quad 2x-3=0 \quad \text{or} \quad x+4=0.$$

  Each of these linear factors can be solved independently.

$$x=-3/2 \quad \text{or} \quad x=3/2 \quad \text{or} \quad x=-4$$

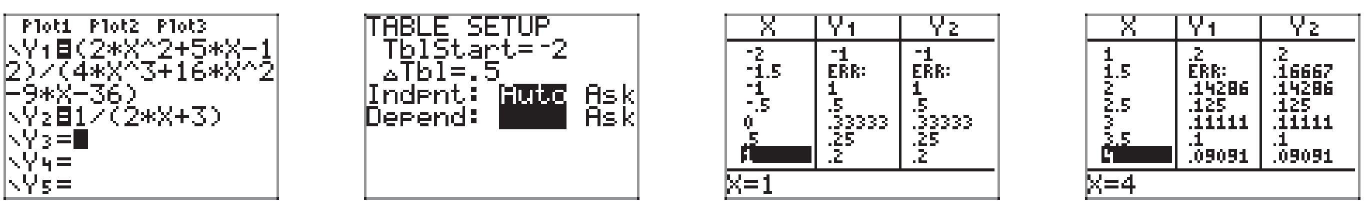
  Each of these x-values is a restriction.
- In the final rational expression, the denominator is $2x+3$. This expression equals zero when $x=-3/2$ and provides no new restrictions.
- Because the denominator of the original expression, namely $4x^3+16x^2-9x-36$, is identical to its factored form in the body our work, this denominator will produce no new restrictions.

Thus, for all values of x,

$$\frac{2x^2+5x-12}{4x^3+16x^2-9x-36}=\frac{1}{2x+3}, \tag{18}$$

provided $x \neq -3/2$, $3/2$, or -4. These are the restrictions. The two expressions are identical for all other values of x.

Finally, let's check this result with our graphing calculator. Load each side of **equation (18)** into the `Y=` menu, as shown in **Figure 2**(a). We know that we have a restriction at $x=-3/2$, so let's set `TblStart` $=-2$ and Δ`Tbl` $=0.5$, as shown in **Figure 2**(b). Be sure that you have `AUTO` set for both independent and dependent variables. Push the `TABLE` button to produce the tabular display shown in **Figure 2**(c).



(a) (b) (c) (d)

Figure 2. Using the graphing calculator to check that the left- and right-hand sides of **equation (18)** are identical.

Remember that we placed the left- and right-hand sides of **equation (18)** in `Y1` and `Y2`, respectively.

- In **Figure 2**(c), note that the expressions `Y1` and `Y2` agree at all values of x except $x=-1.5$. This is the restriction $-3/2$ we found above.
- Use the down arrow key to scroll down in the table shown in **Figure 2**(c) to produce the tabular view shown in **Figure 2**(d). Note that `Y1` and `Y2` agree for all values of x except $x=1.5$. This is the restriction $3/2$ we found above.
- We leave it to our readers to uncover the restriction at $x=-4$ by using the up-arrow to scroll up in the table until you reach an x-value of -4. You should uncover

another ERR (error) message at this x-value because it is a restriction. You get the ERR message due to the fact that the denominator of the left-hand side of **equation (18)** is zero at $x = -4$.

Sign Changes

It is not uncommon that you will have to manipulate the signs in a fraction in order to obtain common factors that can be then cancelled. Consider, for example, the rational expression

$$\frac{3-x}{x-3}. \tag{19}$$

One possible approach is to factor -1 out of the numerator to obtain

$$\frac{3-x}{x-3} = \frac{-(x-3)}{x-3}.$$

You can now cancel common factors.[7]

$$\frac{3-x}{x-3} = \frac{-(x-3)}{x-3} = \frac{-\cancel{(x-3)}}{\cancel{x-3}} = -1$$

This result is valid for all values of x, provided $x \neq 3$.

Let's look at another example.

► **Example 20.** *Reduce the rational expression*

$$\frac{2x-2x^3}{3x^3+4x^2-3x-4} \tag{21}$$

to lowest terms. State all restrictions.

In the numerator, factor out $2x$, then complete the factorization using the difference of two squares pattern.

$$2x - 2x^3 = 2x(1-x^2) = 2x(1+x)(1-x)$$

The denominator can be factored by grouping.

$$\begin{aligned} 3x^3+4x^2-3x-4 &= x^2(3x+4) - 1(3x+4) \\ &= (x^2-1)(3x+4) \\ &= (x+1)(x-1)(3x+4) \end{aligned}$$

Note how the difference of two squares pattern was applied in the last step.

[7] When everything cancels, the resulting rational expression equals 1. For example, consider 6/6, which surely is equal to 1. If we factor and cancel common factors, everything cancels.

$$\frac{6}{6} = \frac{2\cdot 3}{2\cdot 3} = \frac{\cancel{2}\cdot\cancel{3}}{\cancel{2}\cdot\cancel{3}} = 1$$

At this point,

$$\frac{2x-2x^3}{3x^3+4x^2-3x-4}=\frac{2x(1+x)(1-x)}{(x+1)(x-1)(3x+4)}.$$

Because we have $1-x$ in the numerator and $x-1$ in the denominator, we will factor out a -1 from $1-x$, and because the order of factors does not affect their product, we will move the -1 out to the front of the numerator.

$$\frac{2x-2x^3}{3x^3+4x^2-3x-4}=\frac{2x(1+x)(-1)(x-1)}{(x+1)(x-1)(3x+4)}=\frac{-2x(1+x)(x-1)}{(x+1)(x-1)(3x+4)}$$

We can now cancel common factors.

$$\begin{aligned}\frac{2x-2x^3}{3x^3+4x^2-3x-4}&=\frac{-2x(1+x)(x-1)}{(x+1)(x-1)(3x+4)}\\&=\frac{-2x\cancel{(1+x)}\cancel{(x-1)}}{\cancel{(x+1)}\cancel{(x-1)}(3x+4)}\\&=\frac{-2x}{3x+4}\end{aligned}$$

Note that $x+1$ is identical to $1+x$ and cancels. Thus,

$$\frac{2x-2x^3}{3x^3+4x^2-3x-4}=\frac{-2x}{3x+4} \tag{22}$$

for all values of x, provided $x \neq -1$, 1, or $-4/3$. These are the restrictions, values of x that make denominators equal to zero.

The Sign Change Rule for Fractions

Let's look at an alternative approach to the last example. First, let's share the precept that every fraction has three signs, one on the numerator, one on the denominator, and a third on the fraction bar. Thus,

$$\frac{-2}{3} \qquad \text{has understood signs} \qquad +\frac{-2}{+3}.$$

Let's state the *sign change rule* for fractions.

The Sign Change Rule for Fractions. Every fraction has three signs, one on the numerator, one on the denominator, and one on the fraction bar. If you don't see an explicit sign, then a plus sign is understood. If you negate any two of these parts,

- numerator and denominator, or
- numerator and fraction bar, or
- fraction bar and denominator,

then the fraction remains unchanged.

For example, let's start with $-2/3$, then do two negations: numerator and fraction bar. Then,

$$+\frac{-2}{+3} = -\frac{+2}{+3}, \qquad \text{or with understood plus signs,} \qquad \frac{-2}{3} = -\frac{2}{3}.$$

This is a familiar result, as negative two divided by a positive three equals a negative two-thirds.

On another note, we might decide to negate numerator and denominator. Then $-2/3$ becomes

$$+\frac{-2}{+3} = \frac{+2}{-3}, \qquad \text{or with understood plus signs,} \qquad \frac{-2}{3} = \frac{2}{-3}.$$

Again, a familiar result. Certainly, negative two divided by positive three is the same as positive two divided by negative three. They both equal minus two-thirds.

So there you have it. Negate any two parts of a fraction and it remains unchanged. On the surface, this seems a trivial remark, but it can be put to good use when reducing rational expressions. Suppose, for example, that we take the original rational expression from **Example 20** and negate the numerator and fraction bar.

$$\frac{2x - 2x^3}{3x^3 + 4x^2 - 3x - 4} = -\frac{2x^3 - 2x}{3x^3 + 4x^2 - 3x - 4}$$

Note how we've made two sign changes. We've negated the fraction bar, we've negated the numerator ($-(2x - 2x^3) = 2x^3 - 2x$), and left the denominator alone. Therefore, the fraction is unchanged according to our sign change rule.

Now, factor and cancel common factors (we leave the steps for our readers — they're similar to those we took in **Example 20**).

$$\begin{aligned}\frac{2x - 2x^3}{3x^3 + 4x^2 - 3x - 4} &= -\frac{2x^3 - 2x}{3x^3 + 4x^2 - 3x - 4} \\ &= -\frac{2x(x+1)(x-1)}{(x+1)(x-1)(3x+4)} \\ &= -\frac{2x\cancel{(x+1)}\cancel{(x-1)}}{\cancel{(x+1)}\cancel{(x-1)}(3x+4)} \\ &= -\frac{2x}{3x+4}\end{aligned}$$

But does this answer match the answer in **equation (22)**? It does, as can be seen by making two negations, fraction bar and numerator.

$$-\frac{2x}{3x+4} = \frac{-2x}{3x+4}$$

The Secant Line

Consider the graph of the function f that we've drawn in **Figure 3**. Note that we've chosen two points on the graph of f, namely $(a, f(a))$ and $(x, f(x))$, and we've drawn a line L through them that mathematicians call the "secant line."

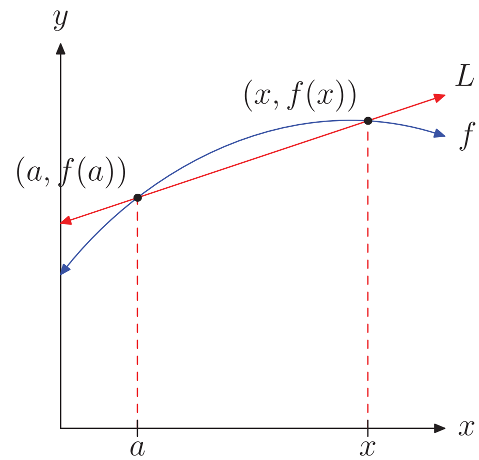


Figure 3. The secant line passes through $(a, f(a))$ and $(x, f(x))$.

The slope of the secant line L is found by dividing the change in y by the change in x.

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a} \tag{23}$$

This slope provides the average rate of change of the variable y with respect to the variable x. Students in calculus use this "average rate of change" to develop the notion of "instantaneous rate of change." However, we'll leave that task for the calculus students and concentrate on the challenge of simplifying the expression **equation (23)** for the average rate of change.

I **Example 24.** *Given the function $f(x) = x^2$, simplify the expression for the average rate of change, namely*

$$\frac{f(x) - f(a)}{x - a}.$$

First, note that $f(x) = x^2$ and $f(a) = a^2$, so we can write

$$\frac{f(x) - f(a)}{x - a} = \frac{x^2 - a^2}{x - a}.$$

We can now use the difference of two squares pattern to factor the numerator and cancel common factors.

$$\frac{x^2 - a^2}{x - a} = \frac{(x + a)\cancel{(x - a)}}{\cancel{x - a}} = x + a$$

Thus,

$$\frac{f(x)-f(a)}{x-a}=x+a,$$

provided, of course, that $x \neq a$.

Let's look at another example.

▶ **Example 25.** *Consider the function* $f(x)=x^2-3x-5$*. Simplify*

$$\frac{f(x)-f(2)}{x-2}.$$

First, $f(x)=x^2-3x-5$ and therefore $f(2)=(2)^2-3(2)-5=-7$, so we can write

$$\frac{f(x)-f(2)}{x-2}=\frac{(x^2-3x-5)-(-7)}{x-2}=\frac{x^2-3x+2}{x-2}.$$

We can now factor the numerator and cancel common factors.

$$\frac{x^2-3x+2}{x-2}=\frac{\cancel{(x-2)}(x-1)}{\cancel{x-2}}=x-1$$

Thus,

$$\frac{f(x)-f(2)}{x-2}=x-1,$$

provided, of course, that $x \neq 2$.

3.2 Exercises

In **Exercises 1-12**, reduce each rational number to lowest terms by applying the following steps:

i. Prime factor both numerator and denominator.
ii. Cancel common prime factors.
iii. Simplify the numerator and denominator of the result.

1. $\dfrac{147}{98}$

2. $\dfrac{3087}{245}$

3. $\dfrac{1715}{196}$

4. $\dfrac{225}{50}$

5. $\dfrac{1715}{441}$

6. $\dfrac{56}{24}$

7. $\dfrac{108}{189}$

8. $\dfrac{75}{500}$

9. $\dfrac{100}{28}$

10. $\dfrac{98}{147}$

11. $\dfrac{1125}{175}$

12. $\dfrac{3087}{8575}$

In **Exercises 13-18**, reduce the given expression to lowest terms. State all restrictions.

13. $\dfrac{x^2 - 10x + 9}{5x - 5}$

14. $\dfrac{x^2 - 9x + 20}{x^2 - x - 20}$

15. $\dfrac{x^2 - 2x - 35}{x^2 - 7x}$

16. $\dfrac{x^2 - 15x + 54}{x^2 + 7x - 8}$

17. $\dfrac{x^2 + 2x - 63}{x^2 + 13x + 42}$

18. $\dfrac{x^2 + 13x + 42}{9x + 63}$

In **Exercises 19-24**, negate any two parts of the fraction, then factor (if necessary) and cancel common factors to reduce the rational expression to lowest terms. State all restrictions.

19. $\dfrac{x + 2}{-x - 2}$

20. $\dfrac{4 - x}{x - 4}$

21. $\dfrac{2x - 6}{3 - x}$

22. $\dfrac{3x + 12}{-x - 4}$

23. $\dfrac{3x^2 + 6x}{-x - 2}$

24. $\dfrac{8x - 2x^2}{x - 4}$

In **Exercises 25-38**, reduce each of the given rational expressions to lowest terms. State all restrictions.

25. $\dfrac{x^2 - x - 20}{25 - x^2}$

26. $\dfrac{x - x^2}{x^2 - 3x + 2}$

27. $\dfrac{x^2 + 3x - 28}{x^2 + 5x - 36}$

28. $\dfrac{x^2 + 10x + 9}{x^2 + 15x + 54}$

29. $\dfrac{x^2 - x - 56}{8x - x^2}$

30. $\dfrac{x^2 - 7x + 10}{5x - x^2}$

31. $\dfrac{x^2 + 13x + 42}{x^2 - 2x - 63}$

32. $\dfrac{x^2 - 16}{x^2 - x - 12}$

33. $\dfrac{x^2 - 9x + 14}{49 - x^2}$

34. $\dfrac{x^2 + 7x + 12}{9 - x^2}$

35. $\dfrac{x^2 - 3x - 18}{x^2 - 6x + 5}$

36. $\dfrac{x^2 + 5x - 6}{x^2 - 1}$

37. $\dfrac{x^2 - 3x - 10}{-9x - 18}$

38. $\dfrac{x^2 - 6x + 8}{16 - x^2}$

In **Exercises 39-42**, reduce each rational function to lowest terms, and then perform each of the following tasks.

i. Load the original rational expression into `Y1` and the reduced rational expression (your answer) into `Y2` of your graphing calculator.
ii. In `TABLE SETUP`, set `TblStart` equal to zero, Δ`Tbl` equal to 1, then make sure both independent and dependent variables are set to `Auto`. Select `TABLE` and scroll with the up- and down-arrows on your calculator until the smallest restriction is in view. Copy both columns of the table onto your homework paper, showing the agreement between `Y1` and `Y2` and what happens at all restrictions.

39. $\dfrac{x^2 - 8x + 7}{x^2 - 11x + 28}$

40. $\dfrac{x^2 - 5x}{x^2 - 9x}$

41. $\dfrac{8x - x^2}{x^2 - x - 56}$

42. $\dfrac{x^2 + 13x + 40}{-2x - 16}$

Given $f(x) = 2x + 5$, simplify each of the expressions in **Exercises 43-46**. Be sure to reduce your answer to lowest terms and state any restrictions.

43. $\dfrac{f(x) - f(3)}{x - 3}$

44. $\dfrac{f(x) - f(6)}{x - 6}$

45. $\dfrac{f(x) - f(a)}{x - a}$

46. $\dfrac{f(a + h) - f(a)}{h}$

Given $f(x) = x^2 + 2x$, simplify each of the expressions in **Exercises 47-50**. Be sure to reduce your answer to lowest terms and state any restrictions.

47. $\dfrac{f(x) - f(1)}{x - 1}$

48. $\dfrac{f(x) - f(a)}{x - a}$

49. $\dfrac{f(a + h) - f(a)}{h}$

50. $\dfrac{f(x + h) - f(x)}{h}$

Drill for Skill. In **Exercises 51-54**, evaluate the given function at the given expression and simplify your answer.

51. Suppose that f is the function

$$f(x) = -\frac{x - 6}{8x + 7}.$$

Evaluate $f(-3x + 2)$ and simplify your answer.

52. Suppose that f is the function

$$f(x) = -\frac{5x + 3}{7x + 6}.$$

Evaluate $f(-5x + 1)$ and simplify your answer.

53. Suppose that f is the function

$$f(x) = -\frac{3x - 6}{4x + 6}.$$

Evaluate $f(-x-3)$ and simplify your answer.

54. Suppose that f is the function

$$f(x) = \frac{4x - 1}{2x - 4}.$$

Evaluate $f(5x)$ and simplify your answer.

3.2 Answers

1. $\frac{3}{2}$

3. $\frac{35}{4}$

5. $\frac{35}{9}$

7. $\frac{4}{7}$

9. $\frac{25}{7}$

11. $\frac{45}{7}$

13. $\frac{x-9}{5}$, provided $x \neq 1$

15. $\frac{x+5}{x}$, provided $x \neq 0,\ 7$

17. $\frac{(x-7)(x+9)}{(x+7)(x+6)}$, provided $x \neq -7,\ -6$

19. -1, provided $x \neq -2$

21. -2, provided $x \neq 3$

23. $-3x$, provided $x \neq -2$

25. $-\frac{x+4}{x+5}$, provided $x \neq -5,\ 5$

27. $\frac{x+7}{x+9}$, provided $x \neq 4,\ -9$

29. $-\frac{x+7}{x}$, provided $x \neq 0,\ 8$

31. $\frac{x+6}{x-9}$, provided $x \neq -7,\ 9$

33. $-\frac{x-2}{x+7}$, provided $x \neq 7,\ -7$

35. $\frac{(x-6)(x+3)}{(x-1)(x-5)}$, provided $x \neq 1,\ 5$

37. $-\frac{x-5}{9}$, provided $x \neq -2$

39. $\frac{x-1}{x-4}$, provided $x \neq 7,\ 4$

X	Y1	Y2
3	-2	-2
4	Err:	Err:
5	4	4
6	2.5	2.5
7	Err:	2
8	1.75	1.75

41. $-\frac{x}{x+7}$, provided $x \neq -7,\ 8$

X	Y1	Y2
-8	-8	-8
-7	Err:	Err:
-6	6	6
-5	2.5	2.5
-4	1.33333	1.33333
-3	0.75	0.75
-2	0.4	0.4
-1	0.166667	0.166667
0	-0	-0
1	-0.125	-0.125
2	-0.222222	-0.222222
3	-0.3	-0.3
4	-0.363636	-0.363636
5	-0.416667	-0.416667
6	-0.461538	-0.461538
7	-0.5	-0.5
8	Err:	-0.533333
9	-0.5625	-0.5625

43. 2, provided $x \neq 3$

45. 2, provided $x \neq a$

47. $x+3$, provided $x \neq 1$

49. $2a+h+2$, provided $h \neq 0$

51. $-\frac{3x+4}{24x-23}$

53. $-\frac{3x+15}{4x+6}$

3.3 Products and Quotients of Rational Functions

In this section we deal with products and quotients of rational expressions. Before we begin, we'll need to establish some fundamental definitions and technique. We begin with the definition of the product of two rational numbers.

Definition 1. *Let a/b and c/d be rational numbers. The product of these rational numbers is defined by*

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}, \qquad \text{or more compactly,} \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}. \tag{2}$$

The definition simply states that you should multiply the numerators of each rational number to obtain the numerator of the product, and you also multiply the denominators of each rational number to obtain the denominator of the product. For example,

$$\frac{2}{3} \cdot \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7} = \frac{10}{21}.$$

Of course, you should also check to make sure your final answer is reduced to lowest terms.

Let's look at an example.

► **Example 3.** *Simplify the product of rational numbers*

$$\frac{6}{231} \cdot \frac{35}{10}. \tag{4}$$

First, multiply numerators and denominators together as follows.

$$\frac{6}{231} \cdot \frac{35}{10} = \frac{6 \cdot 35}{231 \cdot 10} = \frac{210}{2310}.$$

However, the answer is not reduced to lowest terms. We can express the numerator as a product of primes.

$$210 = 21 \cdot 10 = 3 \cdot 7 \cdot 2 \cdot 5 = 2 \cdot 3 \cdot 5 \cdot 7$$

It's not necessary to arrange the factors in ascending order, but every little bit helps. The denominator can also be expressed as a product of primes.

$$2310 = 10 \cdot 231 = 2 \cdot 5 \cdot 7 \cdot 33 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$$

We can now cancel common factors.

$$\frac{210}{2310} = \frac{2 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11} = \frac{\cancel{2} \cdot \cancel{3} \cdot \cancel{5} \cdot \cancel{7}}{\cancel{2} \cdot \cancel{3} \cdot \cancel{5} \cdot \cancel{7} \cdot 11} = \frac{1}{11} \tag{5}$$

However, this approach is not the most efficient way to proceed, as multiplying numerators and denominators allows the products to grow to larger numbers, as in 210/2310. It is then a little bit harder to prime factor the larger numbers.

A better approach is to factor the smaller numerators and denominators immediately, as follows.

$$\frac{6}{231} \cdot \frac{35}{10} = \frac{2 \cdot 3}{3 \cdot 7 \cdot 11} \cdot \frac{5 \cdot 7}{2 \cdot 5}$$

We could now multiply numerators and denominators, then cancel common factors, which would match identically the last computation in **equation (5)**.

However, we can also employ the following cancellation rule.

Cancellation Rule. When working with the product of two or more rational expressions, factor all numerators and denominators, then cancel. The cancellation rule is simple: cancel a factor "on the top" for an identical factor "on the bottom." Speaking more technically, cancel any factor in any numerator for an identical factor in any denominator.

Thus, we can finish our computation by canceling common factors, canceling "something on the top for something on the bottom."

$$\frac{6}{231} \cdot \frac{35}{10} = \frac{2 \cdot 3}{3 \cdot 7 \cdot 11} \cdot \frac{5 \cdot 7}{2 \cdot 5} = \frac{\cancel{2} \cdot \cancel{3}}{\cancel{3} \cdot \cancel{7} \cdot 11} \cdot \frac{\cancel{5} \cdot \cancel{7}}{\cancel{2} \cdot \cancel{5}} = \frac{1}{11}$$

Note that we canceled a 2, 3, 5, and a 7 "on the top" for a 2, 3, 5, and 7 "on the bottom."[12]

Thus, we have two choices when multiplying rational expressions:

- Multiply numerators and denominators, factor, then cancel.
- Factor numerators and denominators, cancel, then multiply numerators and denominators.

It is the latter approach that we will use in this section. Let's look at another example.

[12] Students will sometimes use the phrase "cross-cancel" when working with the product of rational expressions. Unfortunately, this term implies that cancellation can occur only in a diagonal direction, which is far from the truth. We like to tell our students that there is no such term as "cross-cancel." There is only "cancel," and the rule is: cancel something on the top for something on the bottom, which is vernacular for "cancel a factor from any numerator and the identical factor from any denominator."

▶ **Example 6.** *Simplify the expression*

$$\frac{x^2-x-6}{x^2+2x-15}\cdot\frac{x^2-x-30}{x^2-2x-8} \tag{7}$$

State restrictions.

Use the *ac*-test to factor each numerator and denominator. Then cancel as shown.

$$\begin{aligned}\frac{x^2-x-6}{x^2+2x-15}\cdot\frac{x^2-x-30}{x^2-2x-8} &= \frac{(x+2)(x-3)}{(x-3)(x+5)}\cdot\frac{(x+5)(x-6)}{(x+2)(x-4)}\\ &= \frac{\cancel{(x+2)}\cancel{(x-3)}}{\cancel{(x-3)}\cancel{(x+5)}}\cdot\frac{\cancel{(x+5)}(x-6)}{\cancel{(x+2)}(x-4)}\\ &= \frac{x-6}{x-4}\end{aligned}$$

The first fraction's denominator has factors $x-3$ and $x+5$. Hence, $x=3$ or $x=-5$ will make this denominator zero. Therefore, the 3 and -5 are restrictions.

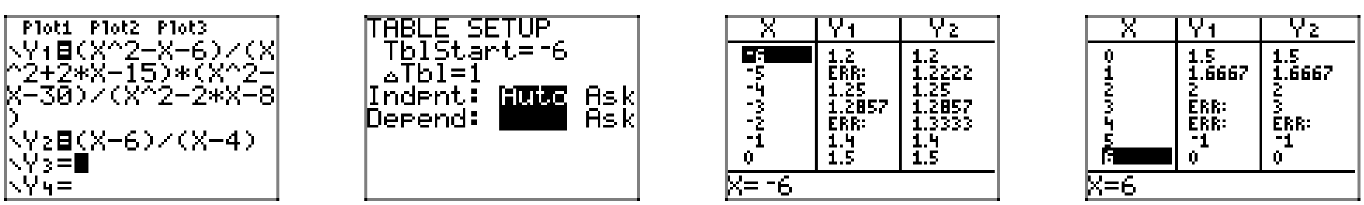
The second fraction's denominator has factors $x+2$ and $x-4$. Hence, $x=-2$ or $x=4$ will make this denominator zero. Therefore, -2 and 4 are restrictions.

Therefore, for all values of x, except the restrictions -5, -2, 3, and 4, the left side of

$$\frac{x^2-x-6}{x^2+2x-15}\cdot\frac{x^2-x-30}{x^2-2x-8}=\frac{x-6}{x-4} \tag{8}$$

is identical to its right side.

It's possible to use your graphing calculator to check your results. First, load the left- and right-hand sides of **equation (8)** into the calculator's into `Y1` and `Y2` in your graphing calculator's `Y=` menu, as shown in **Figure 1**(a). Press `2nd` `TBLSET` and set `TblStart` $=-6$ and Δ`Tbl` $=1$, as shown in **Figure 1**(b). Make sure that `AUTO` is highlighted and selected with the `ENTER` key on both the independent and dependent variables. Press `2nd` `TABLE` to produce the tabular display in **Figure 1**(c).



(a) (b) (c) (d)

Figure 1. Using the table features of the graphing calculator to check the result in **equation (8)**.

Remember that the left- and right-hand sides of **equation (8)** are loaded in `Y1` and `Y2`, respectively.

- In **Figure 1**(c), note the `ERR` (error) message at the restricted values of $x = -5$ and $x = -2$. However, other than at these two restrictions, the functions `Y1` and `Y2` agree at all other values of x in **Figure 1**(c).
- Use the down arrow to scroll down in the table to produce the tabular results shown in **Figure 1**(d). Note the `ERR` (error) message at the restricted values of $x = 3$ and $x = 4$. However, other than at these two restrictions, the functions `Y1` and `Y2` agree at all other values of x in **Figure 1**(d).
- If you scroll up or down in the table, you'll find that the functions `Y1` and `Y2` agree at all values of x other than the restricted values -5, -2, 3, and 4.

Let's look at another example.

Example 9. *Simplify*

$$\frac{9-x^2}{x^2+3x}\cdot\frac{6x-2x^2}{x^2-6x+9} \tag{10}$$

State any restrictions.

The first numerator can be factored using the difference of two squares pattern.

$$9-x^2=(3+x)(3-x).$$

The second denominator is a perfect square trinomial and can be factored as the square of a binomial.

$$x^2-6x+9=(x-3)^2$$

You will want to remove the greatest common factor from the first denominator and second numerator.

$$x^2+3x=x(x+3) \qquad \text{and} \qquad 6x-2x^2=2x(3-x)$$

Thus,

$$\frac{9-x^2}{x^2+3x}\cdot\frac{6x-2x^2}{x^2-6x+9}.=\frac{(3+x)(3-x)}{x(x+3)}\cdot\frac{2x(3-x)}{(x-3)^2}.$$

We'll need to execute a sign change or two to create common factors in the numerators and denominators. So, in both the first and second numerator, factor a -1 from the factor $3-x$ to obtain $3-x=-1(x-3)$. Because the order of factors in a product doesn't matter, we'll slide the -1 to the front in each case.

$$\frac{9-x^2}{x^2+3x}\cdot\frac{6x-2x^2}{x^2-6x+9}.=\frac{-(3+x)(x-3)}{x(x+3)}\cdot\frac{-2x(x-3)}{(x-3)^2}.$$

We can now cancel common factors.

$$\begin{aligned}
\frac{9-x^2}{x^2+3x}\cdot\frac{6x-2x^2}{x^2-6x+9} &= \frac{-(3+x)(x-3)}{x(x+3)}\cdot\frac{-2x(x-3)}{(x-3)^2}\\
&= \frac{-\cancel{(3+x)}\cancel{(x-3)}}{\cancel{x}\cancel{(x+3)}}\cdot\frac{-2\cancel{x}\cancel{(x-3)}}{\cancel{(x-3)^2}}\\
&= 2
\end{aligned}$$

A few things to notice:

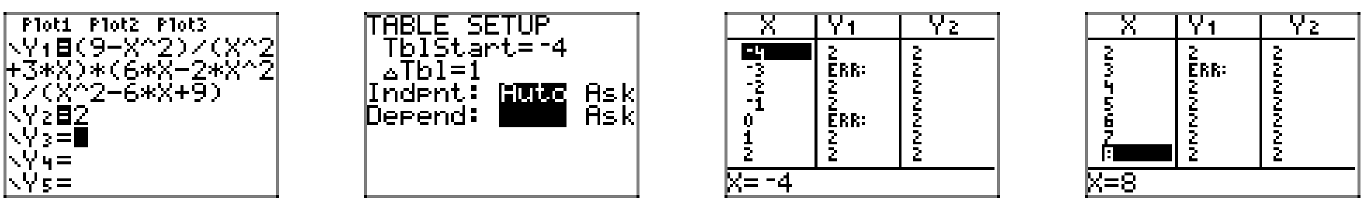
- The factors $3+x$ and $x+3$ are identical, so they may be cancelled, one on the top for one on the bottom.
- Two factors of $x-3$ on the top are cancelled for $(x-3)^2$ (which is equivalent to $(x-3)(x-3)$) on the bottom.
- An x on top cancels an x on the bottom.
- We're left with two minus signs (two -1's) and a 2. So the solution is a positive 2.

Finally, the first denominator has factors x and $x+3$, so $x=0$ and $x=-3$ are restrictions (they make this denominator equal to zero). The second denominator has two factors of $x-3$, so $x=3$ is an additional restriction.

Hence, for all values of x, except the restricted values -3, 0, and 3, the left-hand side of

$$\frac{9-x^2}{x^2+3x}\cdot\frac{6x-2x^2}{x^2-6x+9} = 2 \tag{11}$$

is identical to the right-hand side. Again, this claim is easily tested on the graphing calculator which is evidenced in the sequence of screen captures in **Figure 2**.



(a) (b) (c) (d)

Figure 2. Using the table features of the graphing calculator to check the result in **equation (11)**.

An alternate approach to the problem in **equation (10)** is to note differing orders in the numerators and denominators (descending, ascending powers of x) and *anticipate* the need for a sign change. That is, make the sign change before you factor.

For example, negate (multiply by -1) both numerator and fraction bar of the first fraction to obtain

$$\frac{9-x^2}{x^2+3x} = -\frac{x^2-9}{x^2+3x}$$

According to our sign change rule, negating any two parts of a fraction leaves the fraction unchanged.

If we perform a similar sign change on the second fraction (negate numerator and fraction bar), then we can factor and cancel common factors.

$$\begin{aligned}\frac{9-x^2}{x^2+3x}\cdot\frac{6x-2x^2}{x^2-6x+9} &= -\frac{x^2-9}{x^2+3x}\cdot-\frac{2x^2-6x}{x^2-6x+9}\\ &= -\frac{(x+3)(x-3)}{x(x+3)}\cdot-\frac{2x(x-3)}{(x-3)^2}\\ &= -\frac{\cancel{(x+3)}\cancel{(x-3)}}{\cancel{x}\cancel{(x+3)}}\cdot-\frac{2\cancel{x}\cancel{(x-3)}}{\cancel{(x-3)^2}}\\ &= 2\end{aligned}$$

Division of Rational Expressions

A simple definition will change a problem involving division of two rational expressions into one involving multiplication of two rational expressions. Then there's nothing left to explain, for we already know how to multiply two rational expressions.

So, let's motivate our definition of division. Suppose we ask the question, how many halves are in a whole? The answer is easy, as two halves make a whole. Thus, when we divide 1 by 1/2, we should get 2. There are two halves in one whole.

Let's raise the stakes a bit and ask how many halves are in six? To make the problem more precise, imagine you've ordered 6 pizzas and you cut each in half. How many halves do you have? Again, this is easy when you think about the problem in this manner, the answer is 12. Thus,

$$6 \div \frac{1}{2}$$

(how many halves are in six) is identical to

$$6 \cdot 2,$$

which, of course, is 12. Hopefully, thanks to this opening motivation, the following definition will not seem too strange.

Definition 12. *To perform the division*

$$\frac{a}{b} \div \frac{c}{d},$$

invert the second fraction and multiply, as in

$$\frac{a}{b} \cdot \frac{d}{c}.$$

Thus, if we want to know how many halves are in 12, we change the division into multiplication ("invert and multiply").

$$12 \div \frac{1}{2} = 12 \cdot 2 = 24$$

This makes sense, as there are 24 "halves" in 12. Let's look at a harder example.

▶ **Example 13.** *Simplify*

$$\frac{33}{15} \div \frac{14}{10}. \tag{14}$$

Invert the second fraction and multiply. After that, all we need to do is factor numerators and denominators, then cancel common factors.

$$\frac{33}{15} \div \frac{14}{10} = \frac{33}{15} \cdot \frac{10}{14} = \frac{3 \cdot 11}{3 \cdot 5} \cdot \frac{2 \cdot 5}{2 \cdot 7} = \frac{\cancel{3} \cdot 11}{\cancel{3} \cdot \cancel{5}} \cdot \frac{\cancel{2} \cdot \cancel{5}}{\cancel{2} \cdot 7} = \frac{11}{7}$$

An interesting way to check this result on your calculator is shown in the sequence of screens in **Figure 3**.

(a) (b) (c)

Figure 3. Using the calculator to check division of fractions.

After entering the original problem in your calculator, press `ENTER`, then press the `MATH` button, then select `1:▶ Frac` from the menu and press `ENTER`. The result is shown in **Figure 3**(c), which agrees with our calculation above.

Let's look at another example.

▶ **Example 15.** *Simplify*

$$\frac{9+3x-2x^2}{x^2-16} \div \frac{4x^3-9x}{2x^2+5x-12}. \tag{16}$$

State the restrictions.

Note the order of the first numerator differs from the other numerators and denominators, so we "anticipate" the need for a sign change, negating the numerator and fraction bar of the first fraction. We also invert the second fraction and change the division to multiplication ("invert and multiply").

$$-\frac{2x^2-3x-9}{x^2-16} \cdot \frac{2x^2+5x-12}{4x^3-9x} \tag{17}$$

The numerator in the first fraction in **equation (17)** is a quadratic trinomial, with $ac = (2)(-9) = -18$. The integer pair 3 and -6 has product -18 and sum -3. Hence,

$$\begin{aligned} 2x^2 - 3x - 9 &= 2x^2 + 3x - 6x - 9 \\ &= x(2x+3) - 3(2x+3) \\ &= (x-3)(2x+3). \end{aligned}$$

The denominator of the first fraction in **equation (17)** easily factors using the difference of two squares pattern.

$$x^2 - 16 = (x+4)(x-4)$$

The numerator of the second fraction in **equation (17)** is a quadratic trinomial, with $ac = (2)(-12) = -24$. The integer pair -3 and 8 have product -24 and sum 5. Hence,

$$\begin{aligned} 2x^2 + 5x - 12 &= 2x^2 - 3x + 8x - 12 \\ &= x(2x-3) + 4(2x-3) \\ &= (x+4)(2x-3). \end{aligned}$$

To factor the denominator of the last fraction in **equation (17)**, first pull the greatest common factor (in this case x), then complete the factorization using the difference of two squares pattern.

$$4x^3 - 9x = x(4x^2 - 9) = x(2x+3)(2x-3)$$

We can now replace each numerator and denominator in **equation (17)** with its factorization, then cancel common factors.

$$\begin{aligned} -\frac{2x^2 - 3x - 9}{x^2 - 16} \cdot \frac{2x^2 + 5x - 12}{4x^3 - 9x} &= -\frac{(x-3)(2x+3)}{(x+4)(x-4)} \cdot \frac{(x+4)(2x-3)}{x(2x+3)(2x-3)} \\ &= -\frac{(x-3)\cancel{(2x+3)}}{\cancel{(x+4)}(x-4)} \cdot \frac{\cancel{(x+4)}\cancel{(2x-3)}}{x\cancel{(2x+3)}\cancel{(2x-3)}} \\ &= -\frac{x-3}{x(x-4)} \end{aligned}$$

The last denominator has factors x and $x-4$, so $x=0$ and $x=4$ are restrictions. In the body of our work, the first fraction's denominator has factors $x+4$ and $x-4$. We've seen the factor $x-4$ already, so only the factor $x+4$ adds a new restriction, $x=-4$. Again, in the body of our work, the second fraction's denominator has factors x, $2x+3$, and $2x-3$, so we have added restrictions $x=0$, $x=-3/2$, and $x=3/2$.

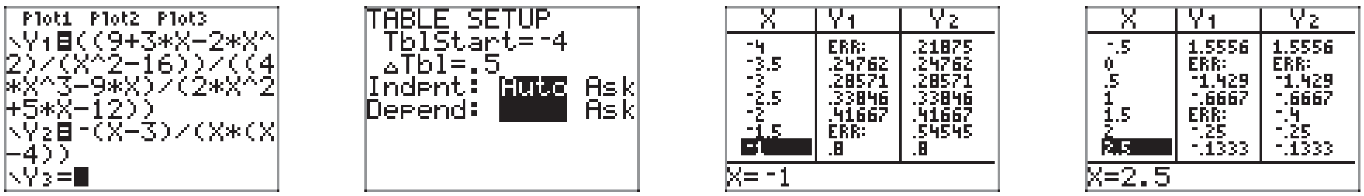
There's one bit of trickery here that can easily be overlooked. In the body of our work, the second fraction's numerator was originally a denominator before we inverted the fraction. So, we must consider what makes this numerator zero as well. Fortunately, the factors in this numerator are $x+4$ and $2x-3$ and we've already considered the restrictions produced by these factors.

Hence, for all values of x, except the restricted values -4, $-3/2$, 0, $3/2$, and 4, the left-hand side of

$$\frac{9 + 3x - 2x^2}{x^2 - 16} \div \frac{4x^3 - 9x}{2x^2 + 5x - 12} = -\frac{x-3}{x(x-4)} \tag{18}$$

is identical to the right-hand side.

Again, this claim is easily checked by using a graphing calculator, as is partially evidenced (you'll have to scroll downward to see the last restriction come into view) in the sequence of screen captures in **Figure 4**.



(a) (b) (c) (d)

Figure 4. Using the table feature of the calculator to check the result in **equation (18)**.

Alternative Notation. Note that the fractional expression a/b means "a divided by b," so we can use this equivalent notation for $a \div b$. For example, the expression

$$\frac{9+3x-2x^2}{x^2-16} \div \frac{4x^3-9x}{2x^2+5x-12} \tag{19}$$

is equivalent to the expression

$$\frac{\dfrac{9+3x-2x^2}{x^2-16}}{\dfrac{4x^3-9x}{2x^2+5x-12}}. \tag{20}$$

Let's look at an example of this notation in use.

Example 21. *Given that*

$$f(x)=\frac{x}{x+3} \qquad \text{and} \qquad g(x)=\frac{x^2}{x+3},$$

simplify both $f(x)g(x)$ and $f(x)/g(x)$.

First, the multiplication. There is no possible cancellation, so we simply multiply numerators and denominators.

$$f(x)g(x)=\frac{x}{x+3}\cdot\frac{x^2}{x+3}=\frac{x^3}{(x+3)^2}.$$

This result is valid for all values of x except -3.

On the other hand,

$$\frac{f(x)}{g(x)}=\frac{\dfrac{x}{x+3}}{\dfrac{x^2}{x+3}}=\frac{x}{x+3}\div\frac{x^2}{x+3}.$$

When we "invert and multiply," then cancel, we obtain

$$\frac{f(x)}{g(x)} = \frac{x}{x+3} \cdot \frac{x+3}{x^2} = \frac{1}{x}.$$

This result is valid for all values of x except -3 and 0.

3.3 *Exercises*

In **Exercises 1-10**, reduce the product to a single fraction in lowest terms.

1. $\frac{108}{14} \cdot \frac{6}{100}$

2. $\frac{75}{63} \cdot \frac{18}{45}$

3. $\frac{189}{56} \cdot \frac{12}{27}$

4. $\frac{45}{72} \cdot \frac{63}{64}$

5. $\frac{15}{36} \cdot \frac{28}{100}$

6. $\frac{189}{49} \cdot \frac{32}{25}$

7. $\frac{21}{100} \cdot \frac{125}{16}$

8. $\frac{21}{35} \cdot \frac{49}{45}$

9. $\frac{56}{20} \cdot \frac{98}{32}$

10. $\frac{27}{125} \cdot \frac{4}{12}$

In **Exercises 11-34**, multiply and simplify. State all restrictions.

11.

$$\frac{x+6}{x^2+16x+63} \cdot \frac{x^2+7x}{x+4}$$

12.

$$\frac{x^2+9x}{x^2-25} \cdot \frac{x^2-x-20}{-18-11x-x^2}$$

13.

$$\frac{x^2+7x+10}{x^2-1} \cdot \frac{-9+10x-x^2}{x^2+9x+20}$$

14.

$$\frac{x^2+5x}{x-4} \cdot \frac{x-2}{x^2+6x+5}$$

15.

$$\frac{x^2-5x}{x^2+2x-48} \cdot \frac{x^2+11x+24}{x^2-x}$$

16.

$$\frac{x^2-6x-27}{x^2+10x+24} \cdot \frac{x^2+13x+42}{x^2-11x+18}$$

17.

$$\frac{-x-x^2}{x^2-9x+8} \cdot \frac{x^2-4x+3}{x^2+4x+3}$$

18.

$$\frac{x^2-12x+35}{x^2+2x-15} \cdot \frac{45+4x-x^2}{x^2+x-30}$$

19.

$$\frac{x+2}{7-x} \cdot \frac{x^2+x-56}{x^2+7x+6}$$

20.

$$\frac{x^2-2x-15}{x^2+x} \cdot \frac{x^2+7x}{x^2+12x+27}$$

21.

$$\frac{x^2-9}{x^2-4x-45} \cdot \frac{x-6}{-3-x}$$

[13] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

22.

$$\frac{x^2-12x+27}{x-4}\cdot\frac{x-5}{x^2-18x+81}$$

23.

$$\frac{x+5}{x^2+12x+32}\cdot\frac{x^2-2x-24}{x+7}$$

24.

$$\frac{x^2-36}{x^2+11x+24}\cdot\frac{-8-x}{x+4}$$

25.

$$\frac{x-5}{x^2-8x+12}\cdot\frac{x^2-12x+36}{x-8}$$

26.

$$\frac{x^2-5x-36}{x-1}\cdot\frac{x-5}{x^2-81}$$

27.

$$\frac{x^2+2x-15}{x^2-10x+16}\cdot\frac{x^2-7x+10}{3x^2+13x-10}$$

28.

$$\frac{5x^2+14x-3}{x+9}\cdot\frac{x-7}{x^2+10x+21}$$

29.

$$\frac{x^2-4}{x^2+2x-63}\cdot\frac{x^2+6x-27}{x^2-6x-16}$$

30.

$$\frac{x^2+5x+6}{x^2-3x}\cdot\frac{x^2-5x}{x^2+9x+18}$$

31.

$$\frac{x-1}{x^2+2x-63}\cdot\frac{x^2-81}{x+4}$$

32.

$$\frac{x^2+9x}{x^2+7x+12}\cdot\frac{27+6x-x^2}{x^2-5x}$$

33.

$$\frac{5-x}{x+3}\cdot\frac{x^2+3x-18}{2x^2-7x-15}$$

34.

$$\frac{4x^2+21x+5}{18-7x-x^2}\cdot\frac{x^2+11x+18}{x^2-25}$$

In **Exercises 35-58**, divide and simplify. State all restrictions.

35.

$$\frac{\dfrac{x^2-14x+48}{x^2+10x+16}}{\dfrac{-24+11x-x^2}{x^2-x-72}}$$

36.

$$\frac{x-1}{x^2-14x+48}\div\frac{x+5}{x^2-3x-18}$$

37.

$$\frac{x^2-1}{x^2-7x+12}\div\frac{x^2+6x+5}{-24+10x-x^2}$$

38.

$$\frac{x^2-13x+42}{x^2-2x-63}\div\frac{x^2-x-42}{x^2+8x+7}$$

39.

$$\frac{x^2-25}{x+1}\div\frac{5x^2+23x-10}{x-3}$$

40.

$$\frac{\dfrac{x^2-3x}{x^2-7x+6}}{\dfrac{x^2-4x}{3x^2-11x-42}}$$

41.

$$\frac{\dfrac{x^2+10x+21}{x-4}}{\dfrac{x^2+3x}{x+8}}$$

42.

$$\frac{x^2+8x+15}{x^2-14x+45} \div \frac{x^2+11x+30}{-30+11x-x^2}$$

43.

$$\frac{\dfrac{x^2-6x-16}{x^2+x-42}}{\dfrac{x^2-64}{x^2+12x+35}}$$

44.

$$\frac{\dfrac{x^2+3x+2}{x^2-9x+18}}{\dfrac{x^2+7x+6}{x^2-6x}}$$

45.

$$\frac{\dfrac{x^2+12x+35}{x+4}}{\dfrac{x^2+10x+25}{x+9}}$$

46.

$$\frac{x^2-8x+7}{x^2+3x-18} \div \frac{x^2-7x}{x^2+6x-27}$$

47.

$$\frac{x^2+x-30}{x^2+5x-36} \div \frac{-6-x}{x+8}$$

48.

$$\frac{\dfrac{2x-x^2}{x^2-15x+54}}{\dfrac{x^2+x}{x^2-11x+30}}$$

49.

$$\frac{\dfrac{x^2-9x+8}{x^2-9}}{\dfrac{x^2-8x}{-15-8x-x^2}}$$

50.

$$\frac{x+5}{x^2+2x+1} \div \frac{x-2}{x^2+10x+9}$$

51.

$$\frac{\dfrac{x^2-4}{x+8}}{\dfrac{x^2-10x+16}{x+3}}$$

52.

$$\frac{27-6x-x^2}{x^2+9x+20} \div \frac{x^2-12x+27}{x^2+5x}$$

53.

$$\frac{\dfrac{x^2+5x+6}{x^2-36}}{\dfrac{x-7}{-6-x}}$$

54.

$$\frac{2-x}{x-5} \div \frac{x^2+3x-10}{x^2-14x+48}$$

55.

$$\frac{\dfrac{x+3}{x^2+4x-12}}{\dfrac{x-4}{x^2-36}}$$

56.

$$\frac{x+3}{x^2-x-2} \div \frac{x}{x^2-3x-4}$$

57.

$$\frac{x^2-11x+28}{x^2+5x+6} \div \frac{7x^2-30x+8}{x^2-x-6}$$

58.

$$\frac{\dfrac{x-7}{3-x}}{\dfrac{2x^2+3x-5}{x^2-12x+27}}$$

59. Let

$$f(x) = \frac{x^2-7x+10}{x^2+4x-21}$$

and

$$g(x) = \frac{5x-x^2}{x^2+15x+56}$$

Compute $f(x)/g(x)$ and simplify your answer.

60. Let

$$f(x) = \frac{x^2+15x+56}{x^2-x-20}$$

and

$$g(x) = \frac{-7-x}{x+1}$$

Compute $f(x)/g(x)$ and simplify your answer.

61. Let

$$f(x) = \frac{x^2+12x+35}{x^2+4x-32}$$

and

$$g(x) = \frac{x^2-2x-35}{x^2+8x}$$

Compute $f(x)/g(x)$ and simplify your answer.

62. Let

$$f(x) = \frac{x^2+4x+3}{x-1}$$

and

$$g(x) = \frac{x^2-4x-21}{x+5}$$

Compute $f(x)/g(x)$ and simplify your answer.

63. Let

$$f(x) = \frac{x^2+x-20}{x}$$

and

$$g(x) = \frac{x-1}{x^2-2x-35}$$

Compute $f(x)g(x)$ and simplify your answer.

64. Let

$$f(x) = \frac{x^2+10x+24}{x^2-13x+42}$$

and

$$g(x) = \frac{x^2-6x-7}{x^2+8x+12}$$

Compute $f(x)g(x)$ and simplify your answer.

65. Let

$$f(x) = \frac{x+5}{-6-x}$$

and

$$g(x) = \frac{x^2+8x+12}{x^2-49}$$

Compute $f(x)g(x)$ and simplify your answer.

66. Let

$$f(x) = \frac{8-7x-x^2}{x^2-8x-9}$$

and

$$g(x) = \frac{x^2-6x-7}{x^2-6x+5}$$

Compute $f(x)g(x)$ and simplify your answer.

3.3 *Answers*

1. $\dfrac{81}{175}$

3. $\dfrac{3}{2}$

5. $\dfrac{7}{60}$

7. $\dfrac{105}{64}$

9. $\dfrac{343}{40}$

11. Provided $x \neq -9, -7, -4$,
$$\frac{x(x+6)}{(x+9)(x+4)}$$

13. Provided $x \neq 1, -1, -4, -5$,
$$-\frac{(x+2)(x-9)}{(x+1)(x+4)}$$

15. Provided $x \neq -8, 6, 1, 0$,
$$\frac{(x-5)(x+3)}{(x-6)(x-1)}$$

17. Provided $x \neq 1, 8, -3, -1$,
$$-\frac{x(x-3)}{(x-8)(x+3)}$$

19. Provided $x \neq 7, -1, -6$,
$$-\frac{(x+2)(x+8)}{(x+1)(x+6)}$$

21. Provided $x \neq -3, -5, 9$,
$$-\frac{(x-3)(x-6)}{(x+5)(x-9)}$$

23. Provided $x \neq -8, -4, -7$,
$$\frac{(x+5)(x-6)}{(x+8)(x+7)}$$

25. Provided $x \neq 2, 6, 8$,
$$\frac{(x-5)(x-6)}{(x-2)(x-8)}$$

27. Provided $x \neq 2, 8, 2/3, -5$,
$$\frac{(x-3)(x-5)}{(3x-2)(x-8)}$$

29. Provided $x \neq -9, 7, 8, -2$,
$$\frac{(x-2)(x-3)}{(x-7)(x-8)}$$

31. Provided $x \neq 7, -9, -4$,
$$\frac{(x-1)(x-9)}{(x-7)(x+4)}$$

33. Provided $x \neq -3, -3/2, 5$,
$$-\frac{(x+6)(x-3)}{(2x+3)(x+3)}$$

35. Provided $x \neq -8, -2, 9, 3, 8$,
$$-\frac{(x-6)(x-9)}{(x+2)(x-3)}$$

37. Provided $x \neq 4, 3, 6, -5, -1$,
$$-\frac{(x-1)(x-6)}{(x-3)(x+5)}$$

39. Provided $x \neq -1, 2/5, -5, 3$,
$$\frac{(x-5)(x-3)}{(5x-2)(x+1)}$$

41. Provided $x \neq 4, 0, -3, -8$,

$$\frac{(x+7)(x+8)}{x(x-4)}$$

43. Provided $x \neq -7, 6, -5, -8, 8$,

$$\frac{(x+2)(x+5)}{(x-6)(x+8)}$$

45. Provided $x \neq -4, -5, -9$,

$$\frac{(x+7)(x+9)}{(x+4)(x+5)}$$

47. Provided $x \neq 4, -9, -8, -6$,

$$-\frac{(x-5)(x+8)}{(x-4)(x+9)}$$

49. Provided $x \neq -3, 3, -5, 0, 8$,

$$-\frac{(x-1)(x+5)}{x(x-3)}$$

51. Provided $x \neq -8, 8, 2, -3$,

$$\frac{(x+2)(x+3)}{(x+8)(x-8)}$$

53. Provided $x \neq 6, -6, 7$,

$$-\frac{(x+2)(x+3)}{(x-6)(x-7)}$$

55. Provided $x \neq 2, -6, 4, 6$,

$$\frac{(x+3)(x-6)}{(x-2)(x-4)}$$

57. Provided $x \neq -2, -3, 3, 2/7, 4$,

$$\frac{(x-7)(x-3)}{(7x-2)(x+3)}$$

59. Provided $x \neq -7, 3, -8, 0, 5$,

$$-\frac{(x-2)(x+8)}{x(x-3)}$$

61. Provided $x \neq -8, 4, 0, 7, -5$,

$$\frac{x(x+7)}{(x-4)(x-7)}$$

63. Provided $x \neq 0, 7, -5$,

$$\frac{(x-4)(x-1)}{x(x-7)}$$

65. Provided $x \neq -6, -7, 7$,

$$-\frac{(x+5)(x+2)}{(x+7)(x-7)}$$

3.4 Sums and Differences of Rational Functions

In this section we concentrate on finding sums and differences of rational expressions. However, before we begin, we need to review some fundamental ideas and technique.

First and foremost is the concept of the multiple of an integer. This is best explained with a simple example. The multiples of 8 is the set of integers $\{8k : k \text{ is an integer}\}$. In other words, if you multiply 8 by 0, ± 1, ± 2, ± 3, ± 4, etc., you produce what is known as the multiples of 8.

$$\text{Multiples of 8 are:}\quad 0,\ \pm 8,\ \pm 16,\ \pm 24,\ \pm 32,\ \text{etc.}$$

However, for our purposes, only the positive multiples are of interest. So we will say:

$$\text{Multiples of 8 are:}\quad 8, 16, \boxed{24}, 32, 40, \boxed{48}, 56, 64, \boxed{72}, \ldots$$

Similarly, we can list the positive multiples of 6.

$$\text{Multiples of 6 are:}\quad 6, 12, 18, \boxed{24}, 30, 36, 42, \boxed{48}, 54, 60, 66, \boxed{72}, \ldots$$

We've framed those numbers that are multiples of both 8 and 6. These are called the *common multiples* of 8 and 6.

$$\text{Common multiples of 8 and 6 are:}\quad 24, 48, 72, \ldots$$

The smallest of this list of common multiples of 8 and 6 is called the *least common multiple* of 8 and 6. We will use the following notation to represent the least common multiple of 8 and 6: $\text{LCM}(8, 6)$.

Hopefully, you will now feel comfortable with the following definition.

Definition 1. *Let a and b be integers. The* **least common multiple** *of a and b, denoted $LCM(a, b)$, is the smallest positive multiple that a and b have in common.*

For larger numbers, listing multiples until you find one in common can be impractical and time consuming. Let's find the least common multiple of 8 and 6 a second time, only this time let's use a different technique.

First, write each number as a product of primes in exponential form.

$$8 = 2^3$$
$$6 = 2 \cdot 3$$

[14] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

Here's the rule.

A Procedure to Find the LCM. To find the LCM of two integers, proceed as follows.

1. Express the prime factorization of each integer in exponential format.
2. To find the least common multiple, write down every prime number that appears, then affix the largest exponent of that prime that appears.

In our example, the primes that occur are 2 and 3. The highest power of 2 that occurs is 2^3. The highest power of 3 that occurs is 3^1. Thus, the LCM(8, 6) is

$$\text{LCM}(8,6) = 2^3 \cdot 3^1 = 24.$$

Note that this result is identical to the result found above by listing all common multiples and choosing the smallest.

Let's try a harder example.

Example 2. *Find the least common multiple of 24 and 36.*

Using the first technique, we list the multiples of each number, framing the multiples in common.

$$\begin{aligned} &\text{Multiples of 24:} \quad 24, 48, \boxed{72}, 96, 120, \boxed{144}, 168, \ldots \\ &\text{Multiples of 36:} \quad 36, \boxed{72}, 108, \boxed{144}, 180 \ldots \end{aligned}$$

The multiples in common are 72, 144, etc., and the least common multiple is $\text{LCM}(24, 36) = 72$.

Now, let's use our second technique to find the least common multiple (LCM). First, express each number as a product of primes in exponential format.

$$\begin{aligned} 24 &= 2^3 \cdot 3 \\ 36 &= 2^2 \cdot 3^2 \end{aligned}$$

To find the least common multiple, write down every prime that occurs and affix the highest power of that prime that occurs. Thus, the highest power of 2 that occurs is 2^3, and the highest power of 3 that occurs is 3^2. Thus, the least common multiple is

$$\text{LCM}(24, 36) = 2^3 \cdot 3^2 = 8 \cdot 9 = 72.$$

Addition and Subtraction Defined

Imagine a pizza that has been cut into 12 equal slices. Then, each slice of pizza represents $1/12$ of the entire pizza.

If Jimmy eats 3 slices, then he has consumed $3/12$ of the entire pizza. If Margaret eats 2 slices, then she has consumed $2/12$ of the entire pizza. It's clear that together they have consumed

$$\frac{3}{12} + \frac{2}{12} = \frac{5}{12}$$

of the pizza. It would seem that adding two fractions with a common denominator is as simple as eating pizza! Hopefully, the following definition will seem reasonable.

Definition 3. *To add two fractions with a common denominator, such as a/c and b/c, add the numerators and divide by the common denominator. In symbols,*

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}.$$

Note how this definition agrees precisely with our pizza consumption discussed above. Here are some examples of adding fractions having common denominators.

$$\begin{aligned} \frac{5}{21} + \frac{3}{21} &= \frac{5+3}{21} \\ &= \frac{8}{21} \end{aligned} \qquad \begin{aligned} \frac{2}{x+2} + \frac{x-3}{x+2} &= \frac{2+(x-3)}{x+2} \\ &= \frac{2+x-3}{x+2} \\ &= \frac{x-1}{x+2} \end{aligned}$$

Subtraction works in much the same way as does addition.

Definition 4. *To subtract two fractions with a common denominator, such as a/c and b/c, subtract the numerators and divide by the common denominator. In symbols,*

$$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}.$$

Here are some examples of subtracting fractions already having common denominators.

$$\begin{aligned} \frac{5}{21} - \frac{3}{21} &= \frac{5-3}{21} \\ &= \frac{2}{21} \end{aligned} \qquad \begin{aligned} \frac{2}{x+2} - \frac{x-3}{x+2} &= \frac{2-(x-3)}{x+2} \\ &= \frac{2-x+3}{x+2} \\ &= \frac{5-x}{x+2} \end{aligned}$$

In the example on the right, note that it is extremely important to use grouping symbols when subtracting numerators. Note that the minus sign in front of the parenthetical expression changes the sign of each term inside the parentheses.

There are times when a sign change will provide a common denominator.

Example 5. *Simplify*

$$\frac{x}{x-3} - \frac{2}{3-x}. \tag{6}$$

State all restrictions.

At first glance, it appears that we do not have a common denominator. On second glance, if we make a sign change on the second fraction, it might help. So, on the second fraction, let's negate the denominator and fraction bar to obtain

$$\frac{x}{x-3} - \frac{2}{3-x} = \frac{x}{x-3} + \frac{2}{x-3} = \frac{x+2}{x-3}.$$

The denominators $x-3$ or $3-x$ are zero when $x = 3$. Hence, 3 is a restricted value. For all other values of x, the left-hand side of

$$\frac{x}{x-3} - \frac{2}{3-x} = \frac{x+2}{x-3}. \tag{7}$$

is identical to the right-hand side.

This is easily tested using the table utility on the graphing calculator, as shown in the sequence of screenshots in **Figure 1**. First load the left- and right-hand sides of **equation (7)** into `Y1` and `Y2` in the `Y=` menu of your graphing calculator, as shown in **Figure 1**(a). Press `2nd TBLSET` and make the changes shown in **Figure 1**(b). Press `2nd TABLE` to produce the table shown in **Figure 1**(c). Note the `ERR` (error) message at the restriction $x = 3$, but note also the agreement of `Y1` and `Y2` for all other values of x.

(a) (b) (c)

Figure 1. Using the table feature of the graphing calculator to check the result in **equation (7)**.

Equivalent Fractions

If you slice a pizza into four equal pieces, then consume two of the four slices, you've consumed half of the pizza. This motivates the fact that

$$\frac{1}{2} = \frac{2}{4}.$$

Indeed, if you slice the pizza into six equal pieces, then consume three slices, you've consumed half of the pizza, so it's fair to say that $3/6 = 1/2$. Indeed, all of the following fractions are equivalent:

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \frac{6}{12} = \frac{7}{14} = \cdots$$

A more formal way to demonstrate that $1/2$ and $7/14$ are equal is to start with the fact that $1/2 = 1/2 \times 1$, then replace 1 with $7/7$ and multiply.

$$\frac{1}{2} = \frac{1}{2} \times 1 = \frac{1}{2} \times \frac{7}{7} = \frac{7}{14}$$

Here's another example of this principle in action, only this time we replace 1 with $(x-2)/(x-2)$.

$$\frac{3}{x+2} = \frac{3}{x+2} \cdot 1 = \frac{3}{x+2} \cdot \frac{x-2}{x-2} = \frac{3(x-2)}{(x+2)(x-2)}$$

In the next example we replace 1 with $(x(x-3))/(x(x-3))$.

$$\frac{2}{x-4} = \frac{2}{x-4} \cdot 1 = \frac{2}{x-4} \cdot \frac{x(x-3)}{x(x-3)} = \frac{2x(x-3)}{x(x-4)(x-3)}$$

Now, let's apply the concept of equivalent fractions to add and subtract fractions with different denominators.

Adding and Subtracting Fractions with Different Denominators

In this section we show our readers how to add and subtract fractions having different denominators. For example, suppose we are asked to add the following fractions.

$$\frac{5}{12} + \frac{5}{18} \tag{8}$$

First, we must find a "common denominator." Fortunately, the machinery to find the "common denominator" is already in place. It turns out that the *least common denominator* for 12 and 18 is the *least common multiple* of 12 and 18.

$$\begin{aligned} 18 &= 2 \cdot 3^2 \\ 12 &= 2^2 \cdot 3 \\ \text{LCD}(12, 18) &= 2^2 \cdot 3^2 = 36 \end{aligned}$$

The next step is to create equivalent fractions using the LCD as the denominator. So, in the case of 5/12,

$$\frac{5}{12} = \frac{5}{12} \cdot 1 = \frac{5}{12} \cdot \frac{3}{3} = \frac{15}{36}.$$

In the case of 5/18,

$$\frac{5}{18} = \frac{5}{18} \cdot 1 = \frac{5}{18} \cdot \frac{2}{2} = \frac{10}{36}.$$

If we replace the fractions in **equation (8)** with their equivalent fractions, we can then add the numerators and divide by the common denominator, as in

$$\frac{5}{12} + \frac{5}{18} = \frac{15}{36} + \frac{10}{36} = \frac{15+10}{36} = \frac{25}{36}.$$

Let's examine a method of organizing the work that is more compact. Consider the following arrangement, where we've used color to highlight the form of 1 required to convert the fractions to equivalent fractions with a common denominator of 36.

$$\begin{aligned}\frac{5}{12} + \frac{5}{18} &= \frac{5}{12} \cdot \frac{3}{3} + \frac{5}{18} \cdot \frac{2}{2} \\ &= \frac{15}{36} + \frac{10}{36} \\ &= \frac{25}{36}\end{aligned}$$

Let's look at a more complicated example.

Example 9. *Simplify the expression*

$$\frac{x+3}{x+2} - \frac{x+2}{x+3}. \qquad (10)$$

State all restrictions.

The denominators are already factored. If we take each factor that appears to the highest exponential power that appears, our least common denominator is $(x+2)(x+3)$. Our first task is to make equivalent fractions having this common denominator.

$$\begin{aligned}\frac{x+3}{x+2} - \frac{x+2}{x+3} &= \frac{x+3}{x+2} \cdot \frac{x+3}{x+3} - \frac{x+2}{x+3} \cdot \frac{x+2}{x+2} \\ &= \frac{x^2+6x+9}{(x+2)(x+3)} - \frac{x^2+4x+4}{(x+2)(x+3)}\end{aligned}$$

Now, subtract the numerators and divide by the common denominator.

$$\begin{aligned}\frac{x+3}{x+2}-\frac{x+2}{x+3} &= \frac{(x^2+6x+9)-(x^2+4x+4)}{(x+2)(x+3)}\\ &= \frac{x^2+6x+9-x^2-4x-4}{(x+2)(x+3)}\\ &= \frac{2x+5}{(x+2)(x+3)}\end{aligned}$$

Note the use of parentheses when we subtracted the numerators. Note further how the minus sign negates each term in the parenthetical expression that follows the minus sign.

> **Tip 11.** *Always use grouping symbols when subtracting the numerators of fractions.*

In the final answer, the factors $x+2$ and $x+3$ in the denominator are zero when $x=-2$ or $x=-3$. These are the restrictions. No other denominators, in the original problem or in the body of our work, provide additional restrictions.

Thus, for all values of x, except the restricted values -2 and -3, the left-hand side of

$$\frac{x+3}{x+2}-\frac{x+2}{x+3}=\frac{2x+5}{(x+2)(x+3)} \tag{12}$$

is identical to the right-hand side. This claim is easily tested on the graphing calculator which is evidenced in the sequence of screen captures in **Figure 2**. Note the `ERR` (error) message at each restricted value of x in **Figure 2**(c), but also note the agreement of `Y1` and `Y2` for all other values of x.

(a) (b) (c)

Figure 2. Using the table feature of the graphing calculator to check the result in **equation (12)**.

Let's look at another example.

▶ **Example 13.** *Simplify the expression*

$$\frac{4}{x^2+6x+5} - \frac{2}{x^2+8x+15}.$$

State all restrictions.

First, factor each denominator.

$$\frac{4}{x^2+6x+5} - \frac{2}{x^2+8x+15} = \frac{4}{(x+1)(x+5)} - \frac{2}{(x+3)(x+5)}$$

The least common denominator, or least common multiple (LCM), requires that we write down each factor that occurs, then affix the highest power of that factor that occurs. Because all factors in the denominators are raised to an understood power of one, the LCD (least common denominator) or LCM is $(x+1)(x+5)(x+3)$.

Next, we make equivalent fractions having this common denominator.

$$\begin{aligned}\frac{4}{x^2+6x+5} - \frac{2}{x^2+8x+15} &= \frac{4}{(x+1)(x+5)} \cdot \frac{x+3}{x+3} - \frac{2}{(x+3)(x+5)} \cdot \frac{x+1}{x+1} \\ &= \frac{4x+12}{(x+3)(x+5)(x+1)} - \frac{2x+2}{(x+3)(x+5)(x+1)}\end{aligned}$$

Subtract the numerators and divide by the common denominator. Be sure to use grouping symbols, particularly with the minus sign that is in play.

$$\begin{aligned}\frac{4}{x^2+6x+5} - \frac{2}{x^2+8x+15} &= \frac{(4x+12)-(2x+2)}{(x+3)(x+5)(x+1)} \\ &= \frac{4x+12-2x-2}{(x+3)(x+5)(x+1)} \\ &= \frac{2x+10}{(x+3)(x+5)(x+1)}\end{aligned}$$

Finally, we should always make sure that our answer is reduced to lowest terms. With that thought in mind, we factor the numerator in hopes that we can get a common factor to cancel.

$$\begin{aligned}\frac{4}{x^2+6x+5} - \frac{2}{x^2+8x+15} &= \frac{2(x+5)}{(x+3)(x+5)(x+1)} \\ &= \frac{2\cancel{(x+5)}}{(x+3)\cancel{(x+5)}(x+1)} \\ &= \frac{2}{(x+3)(x+1)}\end{aligned}$$

The denominators have factors of $x+3$, $x+5$ and $x+1$, so the restrictions are $x=-3$, $x=-5$, and $x=-1$, respectively. For all other values of x, the left-hand side of

$$\frac{4}{x^2+6x+5} - \frac{2}{x^2+8x+15} = \frac{2}{(x+3)(x+1)} \tag{14}$$

is identical to its right-hand side. Again, this is easily tested using the table feature of the graphing calculator, as shown in the screenshots in **Figure 3**. Again, note the `ERR` (error) messages at each restricted value of x, but also note that `Y1` and `Y2` agree for all other values of x.

(a) (b) (c)

Figure 3. Using the table feature of the graphing calculator to check the result in **equation (14)**.

Let's look at another example.

Example 15. *Simplify the expression*

$$\frac{x-3}{x^2-1}+\frac{1}{x+1}-\frac{1}{1-x}.$$

State all restrictions.

First, factor all denominators.

$$\frac{x-3}{x^2-1}+\frac{1}{x+1}-\frac{1}{1-x}=\frac{x-3}{(x+1)(x-1)}+\frac{1}{x+1}-\frac{1}{1-x}$$

If we're not careful, we might be tempted to take one of each factor and use $(x+1)(x-1)(1-x)$ as a common denominator. However, let's first make two negations of the last of the three fractions on the right, negating the fraction bar and denominator to get

$$\frac{x-3}{x^2-1}+\frac{1}{x+1}-\frac{1}{1-x}=\frac{x-3}{(x+1)(x-1)}+\frac{1}{x+1}+\frac{1}{x-1}.$$

Now we can see that a common denominator of $(x+1)(x-1)$ will suffice. Let's make equivalent fractions with this common denominator.

$$\begin{aligned}\frac{x-3}{x^2-1}+\frac{1}{x+1}-\frac{1}{1-x}&=\frac{x-3}{(x+1)(x-1)}+\frac{1}{x+1}\cdot\frac{x-1}{x-1}+\frac{1}{x-1}\cdot\frac{x+1}{x+1}\\&=\frac{x-3}{(x+1)(x-1)}+\frac{x-1}{(x+1)(x-1)}+\frac{x+1}{(x+1)(x-1)}\end{aligned}$$

Add the numerators and divide by the common denominator. Even though grouping symbols are not as critical in this problem (because of the plus signs), we still think it good practice to use them.

$$\begin{aligned}\frac{x-3}{x^2-1}+\frac{1}{x+1}-\frac{1}{1-x} &= \frac{(x-3)+(x-1)+(x+1)}{(x+1)(x-1)}\\ &= \frac{3x-3}{(x+1)(x-1)}\end{aligned}$$

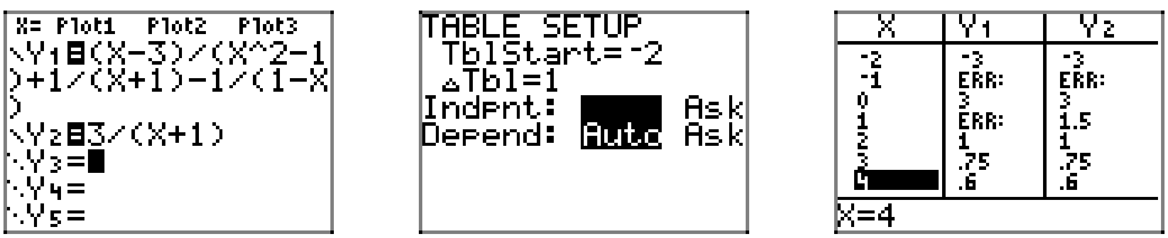
Finally, always make sure that your final answer is reduced to lowest terms. With that thought in mind, we factor the numerator in hopes that we can get a common factor to cancel.

$$\begin{aligned}\frac{x-3}{x^2-1}+\frac{1}{x+1}-\frac{1}{1-x} &= \frac{3(x-1)}{(x+1)(x-1)}\\ &= \frac{3\cancel{(x-1)}}{(x+1)\cancel{(x-1)}}\\ &= \frac{3}{x+1}\end{aligned}$$

The factors $x+1$ and $x-1$ in the denominator produce restrictions $x=-1$ and $x=1$, respectively. However, for all other values of x, the left-hand side of

$$\frac{x-3}{x^2-1}+\frac{1}{x+1}-\frac{1}{1-x}=\frac{3}{x+1} \tag{16}$$

is identical to the right-hand side. Again, this is easily checked on the graphing calculator as shown in the sequence of screenshots in **Figure 4**.



(a) (b) (c)

Figure 4. Using the table feature of the graphing calculator to check the result in **equation (16)**.

Again, note the `ERR` (error) messages at each restriction, but also note that the values of `Y1` and `Y2` agree for all other values of x.

Let's look at an example using function notation.

I **Example 17.** *If the function f and g are defined by the rules*

$$f(x) = \frac{x}{x+2} \qquad \text{and} \qquad g(x) = \frac{1}{x},$$

simplify $f(x) - g(x)$.

First,

$$f(x) - g(x) = \frac{x}{x+2} - \frac{1}{x}.$$

Note how tempting it would be to cancel. However, canceling would be an error in this situation, because subtraction requires a common denominator.

$$\begin{aligned} f(x) - g(x) &= \frac{x}{x+2} \cdot \frac{x}{x} - \frac{1}{x} \cdot \frac{x+2}{x+2} \\ &= \frac{x^2}{x(x+2)} - \frac{x+2}{x(x+2)} \end{aligned}$$

Subtract numerators and divide by the common denominator. This requires that we "distribute" the minus sign.

$$\begin{aligned} f(x) - g(x) &= \frac{x^2 - (x+2)}{x(x+2)} \\ &= \frac{x^2 - x - 2}{x(x+2)} \end{aligned}$$

This result is valid for all values of x except 0 and -2. We leave it to our readers to verify that this result is reduced to lowest terms. You might want to check the result on your calculator as well.

3.4 Exercises

In **Exercises 1-16**, add or subtract the rational expressions, as indicated, and simplify your answer. State all restrictions.

1. $\dfrac{7x^2-49x}{x-6}+\dfrac{42}{x-6}$

2. $\dfrac{2x^2-110}{x-7}-\dfrac{12}{7-x}$

3. $\dfrac{27x-9x^2}{x+3}+\dfrac{162}{x+3}$

4. $\dfrac{2x^2-28}{x+2}-\dfrac{10x}{x+2}$

5. $\dfrac{4x^2-8}{x-4}+\dfrac{56}{4-x}$

6. $\dfrac{4x^2}{x-2}-\dfrac{36x-56}{x-2}$

7. $\dfrac{9x^2}{x-1}+\dfrac{72x-63}{1-x}$

8. $\dfrac{5x^2+30}{x-6}-\dfrac{35x}{x-6}$

9. $\dfrac{4x^2-60x}{x-7}+\dfrac{224}{x-7}$

10. $\dfrac{3x^2}{x-7}-\dfrac{63-30x}{7-x}$

11. $\dfrac{3x^2}{x-2}-\dfrac{48-30x}{2-x}$

12. $\dfrac{4x^2-164}{x-6}-\dfrac{20}{6-x}$

13. $\dfrac{9x^2}{x-2}-\dfrac{81x-126}{x-2}$

14. $\dfrac{9x^2}{x-8}+\dfrac{144x-576}{8-x}$

15. $\dfrac{3x^2-12}{x-3}+\dfrac{15}{3-x}$

16. $\dfrac{7x^2}{x-9}-\dfrac{112x-441}{x-9}$

In **Exercises 17-34**, add or subtract the rational expressions, as indicated, and simplify your answer. State all restrictions.

17. $\dfrac{3x}{x^2-6x+5}+\dfrac{15}{x^2-14x+45}$

18. $\dfrac{7x}{x^2-4x}+\dfrac{28}{x^2-12x+32}$

19. $\dfrac{9x}{x^2+4x-12}-\dfrac{54}{x^2+20x+84}$

20. $\dfrac{9x}{x^2-25}-\dfrac{45}{x^2+20x+75}$

21. $\dfrac{5x}{x^2-21x+98}-\dfrac{35}{7x-x^2}$

22. $\dfrac{7x}{7x-x^2}+\dfrac{147}{x^2+7x-98}$

23. $\dfrac{-7x}{x^2-8x+15}-\dfrac{35}{x^2-12x+35}$

24. $\dfrac{-6x}{x^2+2x}+\dfrac{12}{x^2+6x+8}$

25. $\dfrac{-9x}{x^2-12x+32}-\dfrac{36}{x^2-4x}$

26. $\dfrac{5x}{x^2-12x+32}-\dfrac{20}{4x-x^2}$

27. $\dfrac{6x}{x^2 - 21x + 98} - \dfrac{42}{7x - x^2}$

28. $\dfrac{-2x}{x^2 - 3x - 10} + \dfrac{4}{x^2 + 11x + 18}$

29. $\dfrac{-9x}{x^2 - 6x + 8} - \dfrac{18}{x^2 - 2x}$

30. $\dfrac{6x}{5x - x^2} + \dfrac{90}{x^2 + 5x - 50}$

31. $\dfrac{8x}{5x - x^2} + \dfrac{120}{x^2 + 5x - 50}$

32. $\dfrac{-5x}{x^2 + 5x} + \dfrac{25}{x^2 + 15x + 50}$

33. $\dfrac{-5x}{x^2 + x - 30} + \dfrac{30}{x^2 + 23x + 102}$

34. $\dfrac{9x}{x^2 + 12x + 32} - \dfrac{36}{x^2 + 4x}$

---

35. Let

$$f(x) = \frac{8x}{x^2 + 6x + 8}$$

and

$$g(x) = \frac{16}{x^2 + 2x}$$

Compute $f(x) - g(x)$ and simplify your answer.

36. Let

$$f(x) = \frac{-7x}{x^2 + 8x + 12}$$

and

$$g(x) = \frac{42}{x^2 + 16x + 60}$$

Compute $f(x) + g(x)$ and simplify your answer.

37. Let

$$f(x) = \frac{11x}{x^2 + 12x + 32}$$

and

$$g(x) = \frac{44}{-4x - x^2}$$

Compute $f(x) + g(x)$ and simplify your answer.

38. Let

$$f(x) = \frac{8x}{x^2 - 6x}$$

and

$$g(x) = \frac{48}{x^2 - 18x + 72}$$

Compute $f(x) + g(x)$ and simplify your answer.

39. Let

$$f(x) = \frac{4x}{-x - x^2}$$

and

$$g(x) = \frac{4}{x^2 + 3x + 2}$$

Compute $f(x) + g(x)$ and simplify your answer.

40. Let

$$f(x) = \frac{5x}{x^2 - x - 12}$$

and

$$g(x) = \frac{15}{x^2 + 13x + 30}$$

Compute $f(x) - g(x)$ and simplify your answer.

3.4 *Answers*

1. $7(x-1)$, provided $x \neq 6$.

3. $-9(x-6)$, provided $x \neq -3$.

5. $4(x+4)$, provided $x \neq 4$.

7. $9(x-7)$, provided $x \neq 1$.

9. $4(x-8)$, provided $x \neq 7$.

11. $3(x-8)$, provided $x \neq 2$.

13. $9(x-7)$, provided $x \neq 2$.

15. $3(x+3)$, provided $x \neq 3$.

17. Provided $x \neq 5, 1, 9$,

$$\frac{3(x+1)}{(x-1)(x-9)}$$

19. Provided $x \neq -6, 2, -14$,

$$\frac{9(x+2)}{(x-2)(x+14)}$$

21. Provided $x \neq 7, 14, 0$,

$$\frac{5(x+14)}{x(x-14)}$$

23. Provided $x \neq 5, 3, 7$,

$$\frac{-7(x+3)}{(x-3)(x-7)}$$

25. Provided $x \neq 4, 8, 0$,

$$\frac{-9(x+8)}{x(x-8)}$$

27. Provided $x \neq 7, 14, 0$,

$$\frac{6(x+14)}{x(x-14)}$$

29. Provided $x \neq 2, 4, 0$,

$$\frac{-9(x+4)}{x(x-4)}$$

31. Provided $x \neq 5, 0, -10$,

$$\frac{-8}{x+10}$$

33. Provided $x \neq -6, 5, -17$,

$$\frac{-5(x+5)}{(x-5)(x+17)}$$

35. Provided $x \neq -2, -4, 0$,

$$\frac{8(x-4)}{x(x+4)}$$

37. Provided $x \neq -4, -8, 0$,

$$\frac{11(x-8)}{x(x+8)}$$

39. Provided $x \neq -1, 0, -2$,

$$\frac{-4}{x+2}$$

3.5 Complex Fractions

In this section we learn how to simplify what are called *complex fractions*, an example of which follows.

$$\frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{4}+\frac{2}{3}} \tag{1}$$

Note that both the numerator and denominator are fraction problems in their own right, lending credence to why we refer to such a structure as a "complex fraction."

There are two very different techniques we can use to simplify the complex fraction (**1**). The first technique is a "natural" choice.

> **Simplifying Complex Fractions — First Technique.** To simplify a complex fraction, proceed as follows:
>
> 1. Simplify the numerator.
> 2. Simplify the denominator.
> 3. Simplify the division problem that remains.

Let's follow this outline to simplify the complex fraction (**1**). First, add the fractions in the numerator as follows.

$$\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6} \tag{2}$$

Secondly, add the fractions in the denominator as follows.

$$\frac{1}{4}+\frac{2}{3}=\frac{3}{12}+\frac{8}{12}=\frac{11}{12} \tag{3}$$

Substitute the results from (**2**) and (**3**) into the numerator and denominator of (**1**), respectively.

$$\frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{4}+\frac{2}{3}}=\frac{\frac{5}{6}}{\frac{11}{12}} \tag{4}$$

The right-hand side of (**4**) is equivalent to

$$\frac{5}{6}\div\frac{11}{12}.$$

This is a division problem, so invert and multiply, factor, then cancel common factors.

[16] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

$$\frac{\dfrac{1}{2}+\dfrac{1}{3}}{\dfrac{1}{4}+\dfrac{2}{3}} = \frac{5}{6}\cdot\frac{12}{11}$$
$$= \frac{5}{2\cdot 3}\cdot\frac{2\cdot 2\cdot 3}{11}$$
$$= \frac{5}{\cancel{2}\cdot\cancel{3}}\cdot\frac{\cancel{2}\cdot 2\cdot\cancel{3}}{11}$$
$$= \frac{10}{11}$$

Here is an arrangement of the work, from start to finish, presented without comment. This is a good template to emulate when doing your homework.

$$\frac{\dfrac{1}{2}+\dfrac{1}{3}}{\dfrac{1}{4}+\dfrac{2}{3}} = \frac{\dfrac{3}{6}+\dfrac{2}{6}}{\dfrac{3}{12}+\dfrac{8}{12}}$$
$$= \frac{\dfrac{5}{6}}{\dfrac{11}{12}}$$
$$= \frac{5}{6}\cdot\frac{12}{11}$$
$$= \frac{5}{2\cdot 3}\cdot\frac{2\cdot 2\cdot 3}{11}$$
$$= \frac{5}{\cancel{2}\cdot\cancel{3}}\cdot\frac{\cancel{2}\cdot 2\cdot\cancel{3}}{11}$$
$$= \frac{10}{11}$$

Now, let's look at a second approach to the problem. We saw that simplifying the numerator in (**2**) required a common denominator of 6. Simplifying the denominator in (**3**) required a common denominator of 12. So, let's choose another common denominator, this one a common denominator for both numerator and denominator, namely, 12. Now, multiply top and bottom (numerator and denominator) of the complex fraction (**1**) by 12, as follows.

$$\frac{\dfrac{1}{2}+\dfrac{1}{3}}{\dfrac{1}{4}+\dfrac{2}{3}} = \frac{\left(\dfrac{1}{2}+\dfrac{1}{3}\right)12}{\left(\dfrac{1}{4}+\dfrac{2}{3}\right)12} \tag{5}$$

Distribute the 12 in both numerator and denominator and simplify.

$$\frac{\left(\frac{1}{2}+\frac{1}{3}\right)12}{\left(\frac{1}{4}+\frac{2}{3}\right)12} = \frac{\left(\frac{1}{2}\right)12+\left(\frac{1}{3}\right)12}{\left(\frac{1}{4}\right)12+\left(\frac{2}{3}\right)12} = \frac{6+4}{3+8} = \frac{10}{11}$$

Let's summarize this second technique.

Simplifying Complex Fractions — Second Technique. To simplify a complex fraction, proceed as follows:

1. Find a common denominator for both numerator and denominator.
2. Clear fractions from the numerator and denomaintor by multiplying each by the common denominator found in the first step.

Note that for this particular problem, the second method is much more efficient. It saves both space and time and is more aesthetically pleasing. It is the technique that we will favor in the rest of this section.

Let's look at another example.

I **Example 6.** *Use both the First and Second Techniques to simplify the expression*

$$\frac{\frac{1}{x}-1}{1-\frac{1}{x^2}}. \tag{7}$$

State all restrictions.

Let's use the first technique, simplifying numerator and denominator separately before dividing. First, make equivalent fractions with a common denominator for the subtraction problem in the numerator of (**7**) and simplify. Do the same for the denominator.

$$\frac{\frac{1}{x}-1}{1-\frac{1}{x^2}} = \frac{\frac{1}{x}-\frac{x}{x}}{\frac{x^2}{x^2}-\frac{1}{x^2}} = \frac{\frac{1-x}{x}}{\frac{x^2-1}{x^2}}$$

Next, invert and multiply, then factor.

$$\frac{\frac{1}{x}-1}{1-\frac{1}{x^2}} = \frac{1-x}{x}\cdot\frac{x^2}{x^2-1} = \frac{1-x}{x}\cdot\frac{x^2}{(x+1)(x-1)}$$

Let's invoke the sign change rule and negate two parts of the fraction $(1-x)/x$, numerator and fraction bar, then cancel the common factors.

$$\frac{\frac{1}{x}-1}{1-\frac{1}{x^2}} = -\frac{x-1}{x}\cdot\frac{x^2}{(x+1)(x-1)} = -\frac{\cancel{x-1}}{\cancel{x}}\cdot\frac{x\cancel{x}}{(x+1)\cancel{(x-1)}}$$

Hence,

$$\frac{\frac{1}{x}-1}{1-\frac{1}{x^2}} = -\frac{x}{x+1}.$$

Now, let's try the problem a second time, multiplying numerator and denominator by x^2 to clear fractions from both the numerator and denominator.

$$\frac{\frac{1}{x}-1}{1-\frac{1}{x^2}} = \frac{\left(\frac{1}{x}-1\right)x^2}{\left(1-\frac{1}{x^2}\right)x^2} = \frac{\left(\frac{1}{x}\right)x^2-(1)x^2}{(1)x^2-\left(\frac{1}{x^2}\right)x^2} = \frac{x-x^2}{x^2-1}$$

The order in the numerator of the last fraction intimates that a sign change would be helpful. Negate the numerator and fraction bar, factor, then cancel common factors.

$$\frac{\frac{1}{x}-1}{1-\frac{1}{x^2}} = -\frac{x^2-x}{x^2-1} = -\frac{x(x-1)}{(x+1)(x-1)} = -\frac{x\cancel{(x-1)}}{(x+1)\cancel{(x-1)}} = -\frac{x}{x+1}$$

This is precisely the same answer found with the first technique. To list the restrictions, we must make sure that no values of x make any denominator equal to zero, at the beginning of the problem, in the body of our work, or in the final answer.

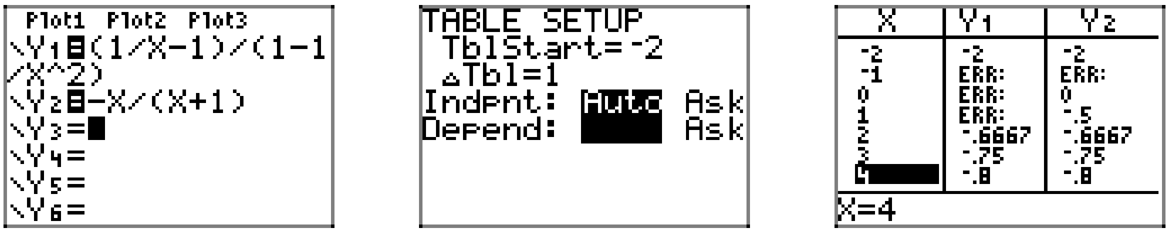
In the original problem, if $x = 0$, then both $1/x$ and $1/x^2$ are undefined, so $x = 0$ is a restriction. In the body of our work, the factors $x + 1$ and $x - 1$ found in various denominators make $x = -1$ and $x = 1$ restrictions. No other denominators supply restrictions that have not already been listed. Hence, for all x other than -1, 0, and 1, the left-hand side of

$$\frac{\frac{1}{x}-1}{1-\frac{1}{x^2}} = -\frac{x}{x+1} \tag{8}$$

is identical to the right-hand side. Again, the calculator's table utility provides ample evidence of this fact in the screenshots shown in **Figure 1**.

Note the ERR (error) messages at each of the restricted values of x, but also note the perfect agreement of Y1 and Y2 at all other values of x.

Let's look at another example, an important example involving function notation.



(a) (b) (c)

Figure 1. Using the table feature of the graphing calculator to check the identity in (8).

► **Example 9.** *Given that*

$$f(x) = \frac{1}{x},$$

simplify the expression

$$\frac{f(x) - f(2)}{x - 2}.$$

List all restrictions.

Remember, $f(2)$ means substitute 2 for x. Because $f(x) = 1/x$, we know that $f(2) = 1/2$, so

$$\frac{f(x) - f(2)}{x - 2} = \frac{\dfrac{1}{x} - \dfrac{1}{2}}{x - 2}.$$

To clear the fractions from the numerator, we'd use a common denominator of $2x$. There are no fractions in the denominator that need clearing, so the common denominator for numerator and denominator is $2x$. Multiply numerator and denominator by $2x$.

$$\frac{f(x) - f(2)}{x - 2} = \frac{\left(\dfrac{1}{x} - \dfrac{1}{2}\right) 2x}{(x - 2)2x} = \frac{\left(\dfrac{1}{x}\right) 2x - \left(\dfrac{1}{2}\right) 2x}{(x - 2)2x} = \frac{2 - x}{2x(x - 2)}$$

Negate the numerator and fraction bar, then cancel common factors.

$$\frac{f(x) - f(2)}{x - 2} = -\frac{x - 2}{2x(x - 2)} = -\frac{\cancel{x - 2}}{2x(\cancel{x - 2})} = -\frac{1}{2x}$$

In the original problem, we have a denominator of $x - 2$, so $x = 2$ is a restriction. If the body of our work, there is a fraction $1/x$, which is undefined when $x = 0$, so $x = 0$ is also a restriction. The remaining denominators provide no other restrictions. Hence, for all values of x except 0 and 2, the left-hand side of

$$\frac{f(x) - f(2)}{x - 2} = -\frac{1}{2x}$$

is identical to the right-hand side.

Let's look at another example involving function notation.

I **Example 10.** *Given*

$$f(x) = \frac{1}{x^2},$$

simplify the expression

$$\frac{f(x+h) - f(x)}{h}. \qquad (11)$$

List all restrictions.

The function notation $f(x+h)$ is asking us to replace each instance of x in the formula $1/x^2$ with $x+h$. Thus, $f(x+h) = 1/(x+h)^2$.

Here is another way to think of this substitution. Suppose that we remove the x from

$$f(x) = \frac{1}{x^2},$$

so that it reads

$$f(\quad) = \frac{1}{(\quad)^2}. \qquad (12)$$

Now, if you want to compute $f(2)$, simply insert a 2 in the blank area between parentheses. In our case, we want to compute $f(x+h)$, so we insert an $x+h$ in the blank space between parentheses in (**12**) to get

$$f(x+h) = \frac{1}{(x+h)^2}.$$

With these preliminary remarks in mind, let's return to the problem. First, we interpret the function notation as in our preliminary remarks and write

$$\frac{f(x+h) - f(x)}{h} = \frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h}.$$

The common denominator for the numerator is found by listing each factor to the highest power that it occurs. Hence, the common denominator is $x^2(x+h)^2$. The denominator has no fractions to be cleared, so it suffices to multiply both numerator and denominator by $x^2(x+h)^2$.

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{\left(\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}\right) x^2(x+h)^2}{hx^2(x+h)^2} \\
&= \frac{\left(\dfrac{1}{(x+h)^2}\right) x^2(x+h)^2 - \left(\dfrac{1}{x^2}\right) x^2(x+h)^2}{hx^2(x+h)^2} \\
&= \frac{x^2 - (x+h)^2}{hx^2(x+h)^2}
\end{aligned}$$

We will now expand the numerator. Don't forget to use parentheses and distribute that minus sign.

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2}\\ &= \frac{x^2-x^2-2xh-h^2}{hx^2(x+h)^2}\\ &= \frac{-2xh-h^2}{hx^2(x+h)^2}\end{aligned}$$

Finally, factor a $-h$ out of the numerator in hopes of finding a common factor to cancel.

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{-h(2x+h)}{hx^2(x+h)^2}\\ &= \frac{-\cancel{h}(2x+h)}{\cancel{h}x^2(x+h)^2}\\ &= \frac{-(2x+h)}{x^2(x+h)^2}\end{aligned}$$

We must now discuss the restrictions. In the original question (**11**), the h in the denominator must not equal zero. Hence, $h=0$ is a restriction. In the final simplified form, the factor of x^2 in the denominator is undefined if $x=0$. Hence, $x=0$ is a restriction. Finally, the factor of $(x+h)^2$ in the final denominator is undefined if $x+h=0$, so $x=-h$ is a restriction. The remaining denominators provide no additional restrictions. Hence, provided $h \neq 0$, $x \neq 0$, and $x \neq -h$, for all other combinations of x and h, the left-hand side of

$$\frac{f(x+h)-f(x)}{h} = \frac{-(2x+h)}{x^2(x+h)^2}$$

is identical to the right-hand side.

Let's look at one final example using function notation.

Example 13. *If*

$$f(x) = \frac{x}{x+1} \tag{14}$$

simplify $f(f(x))$.

We first evaluate f at x, then evaluate f at the result of the first computation. Thus, we work the inner function first to obtain

$$f(f(x)) = f\left(\frac{x}{x+1}\right).$$

The notation $f(x/(x+1))$ is asking us to replace each occurrence of x in the formula $x/(x+1)$ with the expression $x/(x+1)$. Confusing? Here is an easy way to think of this substitution. Suppose that we remove x from

$$f(x)=\frac{x}{x+1},$$

replacing each occurrence of x with empty parentheses, which will produce the template

$$f(\ \ )=\frac{(\ \ )}{(\ \ )+1}. \tag{15}$$

Now, if asked to compute $f(3)$, simply insert 3 into the blank areas between parentheses. In this case, we want to compute $f(x/(x+1))$, so we insert $x/(x+1)$ in the blank space between each set of parentheses in (**15**) to obtain

$$f\left(\frac{x}{x+1}\right)=\frac{\dfrac{x}{x+1}}{\dfrac{x}{x+1}+1}.$$

We now have a complex fraction. The common denominator for both top and bottom of this complex fraction is $x+1$. Thus, we multiply both numerator and denominator of our complex fraction by $x+1$ and use the distributive property as follows.

$$\frac{\dfrac{x}{x+1}}{\dfrac{x}{x+1}+1}=\frac{\left(\dfrac{x}{x+1}\right)(x+1)}{\left(\dfrac{x}{x+1}+1\right)(x+1)}=\frac{\left(\dfrac{x}{x+1}\right)(x+1)}{\left(\dfrac{x}{x+1}\right)(x+1)+(1)(x+1)}$$

Cancel and simplify.

$$\frac{\left(\dfrac{x}{x+1}\right)(x+1)}{\left(\dfrac{x}{x+1}\right)(x+1)+(1)(x+1)}=\frac{x}{x+(x+1)}=\frac{x}{2x+1}$$

In the final denominator, the value $x=-1/2$ makes the denominator $2x+1$ equal to zero. Hence, $x=-1/2$ is a restriction. In the body of our work, several fractions have denominators of $x+1$ and are therefore undefined at $x=-1$. Thus, $x=-1$ is a restriction. No other denominators add additional restrictions.

Hence, for all values of x, except $x=-1/2$ and $x=-1$, the left-hand side of

$$f(f(x))=\frac{x}{2x+1}$$

is identical to the right-hand side.

3.5 *Exercises*

In **Exercises 1-6**, evaluate the function at the given rational number. Then use the first or second technique for simplifying complex fractions explained in the narrative to simplify your answer.

1. Given

$$f(x) = \frac{x+1}{2-x},$$

evaluate and simplify $f(1/2)$.

2. Given

$$f(x) = \frac{2-x}{x+5},$$

evaluate and simplify $f(3/2)$.

3. Given

$$f(x) = \frac{2x+3}{4-x},$$

evaluate and simplify $f(1/3)$.

4. Given

$$f(x) = \frac{3-2x}{x+5},$$

evaluate and simplify $f(2/5)$.

5. Given

$$f(x) = \frac{5-2x}{x+4},$$

evaluate and simplify $f(3/5)$.

6. Given

$$f(x) = \frac{2x-9}{11-x},$$

evaluate and simplify $f(4/3)$.

In **Exercises 7-46**, simplify the given complex rational expression. State all restrictions.

7.

$$\frac{5+\dfrac{6}{x}}{\dfrac{25}{x}-\dfrac{36}{x^3}}$$

8.

$$\frac{7+\dfrac{9}{x}}{\dfrac{49}{x}-\dfrac{81}{x^3}}$$

9.

$$\frac{\dfrac{7}{x-2}-\dfrac{5}{x-7}}{\dfrac{8}{x-7}+\dfrac{3}{x+8}}$$

10.

$$\frac{\dfrac{9}{x+4}-\dfrac{7}{x-9}}{\dfrac{9}{x-9}+\dfrac{5}{x-4}}$$

11.

$$\frac{3+\dfrac{7}{x}}{\dfrac{9}{x^2}-\dfrac{49}{x^4}}$$

12.

$$\frac{2-\frac{5}{x}}{\frac{4}{x^2}-\frac{25}{x^4}}$$

13.

$$\frac{\frac{9}{x+4}+\frac{7}{x+9}}{\frac{9}{x+9}+\frac{2}{x-8}}$$

14.

$$\frac{\frac{4}{x-6}+\frac{9}{x-9}}{\frac{9}{x-6}+\frac{8}{x-9}}$$

15.

$$\frac{\frac{5}{x-7}-\frac{4}{x-4}}{\frac{10}{x-4}-\frac{5}{x+2}}$$

16.

$$\frac{\frac{3}{x+6}+\frac{7}{x+9}}{\frac{9}{x+6}-\frac{4}{x+9}}$$

17.

$$\frac{\frac{6}{x-3}+\frac{5}{x-8}}{\frac{9}{x-3}+\frac{7}{x-8}}$$

18.

$$\frac{\frac{7}{x-7}-\frac{4}{x-2}}{\frac{7}{x-7}-\frac{6}{x-2}}$$

19.

$$\frac{\frac{4}{x-2}+\frac{7}{x-7}}{\frac{5}{x-2}+\frac{2}{x-7}}$$

20.

$$\frac{\frac{9}{x+2}-\frac{7}{x+5}}{\frac{4}{x+2}+\frac{3}{x+5}}$$

21.

$$\frac{5+\frac{4}{x}}{\frac{25}{x}-\frac{16}{x^3}}$$

22.

$$\frac{\frac{6}{x+5}+\frac{5}{x+4}}{\frac{8}{x+5}-\frac{3}{x+4}}$$

23.

$$\frac{\frac{9}{x-5}+\frac{8}{x+4}}{\frac{5}{x-5}-\frac{4}{x+4}}$$

24.

$$\frac{\frac{4}{x-6}+\frac{4}{x-9}}{\frac{6}{x-6}+\frac{6}{x-9}}$$

25.

$$\frac{\frac{6}{x+8}+\frac{5}{x-2}}{\frac{5}{x-2}-\frac{2}{x+2}}$$

26.

$$\frac{\dfrac{7}{x+9}+\dfrac{9}{x-2}}{\dfrac{4}{x-2}+\dfrac{7}{x+1}}$$

27.

$$\frac{\dfrac{7}{x+7}-\dfrac{5}{x+4}}{\dfrac{8}{x+7}-\dfrac{3}{x+4}}$$

28.

$$\frac{25-\dfrac{16}{x^2}}{5+\dfrac{4}{x}}$$

29.

$$\frac{\dfrac{64}{x}-\dfrac{25}{x^3}}{8-\dfrac{5}{x}}$$

30.

$$\frac{\dfrac{4}{x+2}+\dfrac{5}{x-6}}{\dfrac{7}{x-6}-\dfrac{5}{x+7}}$$

31.

$$\frac{\dfrac{2}{x-6}-\dfrac{4}{x+9}}{\dfrac{3}{x-6}-\dfrac{6}{x+9}}$$

32.

$$\frac{\dfrac{3}{x+6}-\dfrac{4}{x+4}}{\dfrac{6}{x+6}-\dfrac{8}{x+4}}$$

33.

$$\frac{\dfrac{9}{x^2}-\dfrac{64}{x^4}}{3-\dfrac{8}{x}}$$

34.

$$\frac{\dfrac{9}{x^2}-\dfrac{25}{x^4}}{3-\dfrac{5}{x}}$$

35.

$$\frac{\dfrac{4}{x-4}-\dfrac{8}{x-7}}{\dfrac{4}{x-7}+\dfrac{2}{x+2}}$$

36.

$$\frac{2-\dfrac{7}{x}}{4-\dfrac{49}{x^2}}$$

37.

$$\frac{\dfrac{3}{x^2+8x-9}+\dfrac{3}{x^2-81}}{\dfrac{9}{x^2-81}+\dfrac{9}{x^2-8x-9}}$$

38.

$$\frac{\dfrac{7}{x^2-5x-14}+\dfrac{2}{x^2-7x-18}}{\dfrac{5}{x^2-7x-18}+\dfrac{8}{x^2-6x-27}}$$

39.

$$\frac{\dfrac{2}{x^2+8x+7}+\dfrac{5}{x^2+13x+42}}{\dfrac{7}{x^2+13x+42}+\dfrac{6}{x^2+3x-18}}$$

40.
$$\frac{\dfrac{3}{x^2+5x-14}+\dfrac{3}{x^2-7x-98}}{\dfrac{3}{x^2-7x-98}+\dfrac{3}{x^2-15x+14}}$$

41.
$$\frac{\dfrac{6}{x^2+11x+24}-\dfrac{6}{x^2+13x+40}}{\dfrac{9}{x^2+13x+40}-\dfrac{9}{x^2-3x-40}}$$

42.
$$\frac{\dfrac{7}{x^2+13x+30}+\dfrac{7}{x^2+19x+90}}{\dfrac{9}{x^2+19x+90}+\dfrac{9}{x^2+7x-18}}$$

43.
$$\frac{\dfrac{7}{x^2-6x+5}+\dfrac{7}{x^2+2x-35}}{\dfrac{8}{x^2+2x-35}+\dfrac{8}{x^2+8x+7}}$$

44.
$$\frac{\dfrac{2}{x^2-4x-12}-\dfrac{2}{x^2-x-30}}{\dfrac{2}{x^2-x-30}-\dfrac{2}{x^2-4x-45}}$$

45.
$$\frac{\dfrac{4}{x^2+6x-7}-\dfrac{4}{x^2+2x-3}}{\dfrac{4}{x^2+2x-3}-\dfrac{4}{x^2+5x+6}}$$

46.
$$\frac{\dfrac{9}{x^2+3x-4}+\dfrac{8}{x^2-7x+6}}{\dfrac{4}{x^2-7x+6}+\dfrac{9}{x^2-10x+24}}$$

47. Given $f(x)=2/x$, simplify
$$\frac{f(x)-f(3)}{x-3}.$$
State all restrictions.

48. Given $f(x)=5/x$, simplify
$$\frac{f(x)-f(2)}{x-2}.$$
State all restrictions.

49. Given $f(x)=3/x^2$, simplify
$$\frac{f(x)-f(1)}{x-1}.$$
State all restrictions.

50. Given $f(x)=5/x^2$, simplify
$$\frac{f(x)-f(2)}{x-2}.$$
State all restrictions.

51. Given $f(x)=7/x$, simplify
$$\frac{f(x+h)-f(x)}{h}.$$
State all restrictions.

52. Given $f(x)=4/x$, simplify
$$\frac{f(x+h)-f(x)}{h}.$$
State all restrictions.

53. Given
$$f(x)=\frac{x+1}{3-x},$$
find and simplify $f(1/x)$. State all restrictions.

54. Given

$$f(x) = \frac{2 - x}{3x + 4},$$

find and simplify $f(2/x)$. State all restrictions.

55. Given

$$f(x) = \frac{x + 1}{2 - 5x},$$

find and simplify $f(5/x)$. State all restrictions.

56. Given

$$f(x) = \frac{2x - 3}{4 + x},$$

find and simplify $f(1/x)$. State all restrictions.

57. Given

$$f(x) = \frac{x}{x + 2},$$

find and simplify $f(f(x))$. State all restrictions.

58. Given

$$f(x) = \frac{2x}{x + 5},$$

find and simplify $f(f(x))$. State all restrictions.

3.5 *Answers*

1. 1

3. 1

5. 19/23

7. Provided $x \neq 0$, $-6/5$, or $6/5$,
$$\frac{x^2}{5x-6}.$$

9. Provided $x \neq 2$, 7, -8, of $-43/11$,
$$\frac{(2x-39)(x+8)}{(11x+43)(x-2)}.$$

11. Provided $x \neq 0$, $-7/3$, or $7/3$,
$$\frac{x^3}{3x-7}.$$

13. Provided $x \neq -4$, -9, 8, or $54/11$,
$$\frac{(16x+109)(x-8)}{(11x-54)(x+4)}.$$

15. Provided $x \neq 7$, 4, -2, or -8,
$$\frac{x+2}{5(x-7)}.$$

17. Provided $x \neq 3$, 8, or $93/16$,
$$\frac{11x-63}{16x-93}.$$

19. Provided $x \neq 2$, 7, or $39/7$,
$$\frac{11x-42}{7x-39}.$$

21. Provided $x \neq 0$, $-4/5$, or $4/5$,
$$\frac{x^2}{5x-4}.$$

23. Provided $x \neq 5$, -4, or -40,
$$\frac{17x-4}{x+40}.$$

25. Provided $x \neq -8$, 2, -2, or $-14/3$,
$$\frac{(11x+28)(x+2)}{(3x+14)(x+8)}.$$

27. Provided $x \neq -7$, -4, or $-11/5$,
$$\frac{2x-7}{5x+11}.$$

29. Provided $x \neq 0$ or $5/8$,
$$\frac{8x+5}{x^2}.$$

31. Provided $x \neq 6$, -9, or 21,
$$\frac{2}{3}.$$

33. Provided $x \neq 0$ or $8/3$,
$$\frac{3x+8}{x^3}.$$

35. Provided $x \neq 4$, 7, -2, or 1,
$$\frac{-2(x+2)}{3(x-4)}.$$

37. Provided $x \neq 1, -9, 9, -1, -5$,
$$\frac{(x-5)(x+1)}{3(x+5)(x-1)}$$

39. Provided $x \neq -1, -7, -6, 3, -21/13$,
$$\frac{(7x+17)(x-3)}{(13x+21)(x+1)}$$

41. Provided $x \neq -3, -8, -5, 8$,

$$\frac{-1(x-8)}{12(x+3)}$$

43. Provided $x \neq 1, 5, -7, -1, 2$,

$$\frac{7(x+3)(x+1)}{8(x-2)(x-1)}$$

45. Provided $x \neq -7, 1, -3, -2$,

$$\frac{-4(x+2)}{3(x+7)}$$

47. Provided $x \neq 0,\ 3$,

$$-\frac{2}{3x}$$

49. Provided $x \neq 0,\ 1$,

$$-\frac{3(x+1)}{x^2}$$

51. Provided $x \neq 0, -h$, and $h \neq 0$,

$$-\frac{7}{x(x+h)}$$

53. Provided $x \neq 0,\ 1/3$,

$$\frac{x+1}{3x-1}$$

55. Provided $x \neq 0,\ 25/2$,

$$\frac{x+5}{2x-25}$$

57. Provided $x \neq -2,\ -4/3$,

$$\frac{x}{3x+4}$$

3.6 Solving Rational Equations[18]

When simplifying complex fractions in the previous section, we saw that multiplying both numerator and denominator by the appropriate expression could "clear" all fractions from the numerator and denominator, greatly simplifying the rational expression.

In this section, a similar technique is used.

Clear the Fractions from a Rational Equation. If your equation has rational expressions, multiply both sides of the equation by the least common denominator to clear the equation of rational expressions.

Let's look at an example.

I **Example 1.** *Solve the following equation for x.*

$$\frac{x}{2} - \frac{2}{3} = \frac{3}{4} \tag{2}$$

To clear this equation of fractions, we will multiply both sides by the common denominator for 2, 3, and 4, which is 12. Distribute 12 in the second step.

$$\begin{aligned} 12\left(\frac{x}{2} - \frac{2}{3}\right) &= \left(\frac{3}{4}\right)12 \\ 12\left(\frac{x}{2}\right) - 12\left(\frac{2}{3}\right) &= \left(\frac{3}{4}\right)12 \end{aligned}$$

Multiply.

$$6x - 8 = 9$$

We've succeeded in clearing the rational expressions from the equation by multiplying through by the common denominator. We now have a simple linear equation which can be solved by first adding 8 to both sides of the equation, followed by dividing both sides of the equation by 6.

$$\begin{aligned} 6x &= 17 \\ x &= \frac{17}{6} \end{aligned}$$

We'll leave it to our readers to check this solution.

Let's try another example.

I **Example 3.** *Solve the following equation for x.*

$$6 = \frac{5}{x} + \frac{6}{x^2} \tag{4}$$

In this equation, the denominators are 1, x, and x^2, and the common denominator for both sides of the equation is x^2. Consequently, we begin the solution by first multiplying both sides of the equation by x^2.

$$\begin{aligned} x^2(6) &= \left(\frac{5}{x} + \frac{6}{x^2}\right) x^2 \\ x^2(6) &= \left(\frac{5}{x}\right) x^2 + \left(\frac{6}{x^2}\right) x^2 \end{aligned}$$

Simplify.

$$6x^2 = 5x + 6$$

Note that multiplying both sides of the original equation by the least common denominator clears the equation of all rational expressions. This last equation is nonlinear,[19] so make one side of the equation equal to zero by subtracting $5x$ and 6 from both sides of the equation.

$$6x^2 - 5x - 6 = 0$$

To factor the left-hand side of this equation, note that it is a quadratic trinomial with $ac = (6)(-6) = -36$. The integer pair 4 and -9 have product -36 and sum -5. Split the middle term using this pair and factor by grouping.

$$\begin{aligned} 6x^2 + 4x - 9x - 6 &= 0 \\ 2x(3x + 2) - 3(3x + 2) &= 0 \\ (2x - 3)(3x + 2) &= 0 \end{aligned}$$

The zero product property forces either

$$2x - 3 = 0 \quad \text{or} \quad 3x + 2 = 0.$$

Each of these linear equations is easily solved.

$$x = \frac{3}{2} \quad \text{or} \quad x = -\frac{2}{3}$$

Of course, we should always check our solutions. Substituting $x = 3/2$ into the right-hand side of the original **equation (4)**,

[19] Whenever an equation in x has a power of x other than 1, the equation is *nonlinear* (the graphs involved are not all lines). As we've seen in previous chapters, the approach to solving a quadratic (second degree) equation should be to make one side of the equation equal to zero, then factor or use the quadratic formula to find the solutions.

$$\frac{5}{x}+\frac{6}{x^2}=\frac{5}{3/2}+\frac{6}{(3/2)^2}=\frac{5}{3/2}+\frac{6}{9/4}.$$

In the final expression, multiply top and bottom of the first fraction by 2, top and bottom of the second fraction by 4.

$$\frac{5}{3/2}\cdot\frac{2}{2}+\frac{6}{9/4}\cdot\frac{4}{4}=\frac{10}{3}+\frac{24}{9}$$

Make equivalent fractions with a common denominator of 9 and add.

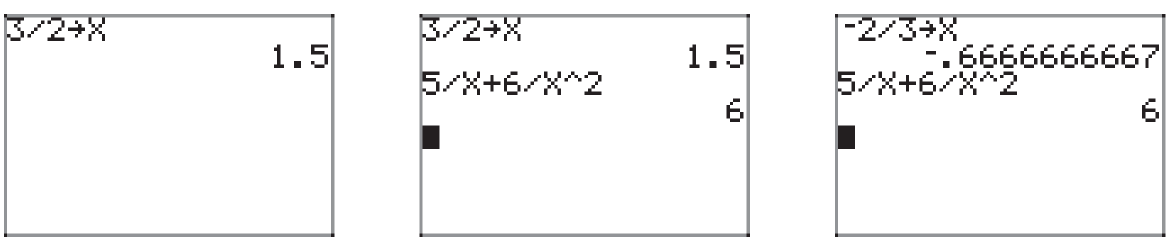
$$\frac{10}{3}\cdot\frac{3}{3}+\frac{24}{9}=\frac{30}{9}+\frac{24}{9}=\frac{54}{9}=6$$

Note that this result is identical to the left-hand side of the original **equation (4)**. Thus, $x = 3/2$ checks.

This example clearly demonstrates that the check can be as difficult and as time consuming as the computation used to originally solve the equation. For this reason, we tend to get lazy and not check our answers as we should. There is help, however, as the graphing calculator can help us check the solutions of equations.

First, enter the solution 3/2 in your calculator screen, push the `STOI` button, then push the `X` button, and execute the resulting command on the screen by pushing the `ENTER` key. The result is shown in **Figure 1**(a).

Next, enter the expression `5/X+6/X^2` and execute the resulting command on the screen by pushing the `ENTER` key. The result is shown in **Figure 1**(b). Note that the result is 6, the same as computed by hand above, and it matches the left-hand side of the original **equation (4)**. We've also used the calculator to check the second solution $x = -2/3$. This is shown in **Figure 4**(c).



(a) (b) (c)

Figure 1. Using the graphing calculator to check the solutions of **equation (4)**.

Let's look at another example.

I **Example 5.** *Solve the following equation for x.*

$$\frac{2}{x^2}=1-\frac{2}{x} \qquad (6)$$

First, multiply both sides of **equation (6)** by the common denominator x^2.

$$x^2\left(\frac{2}{x^2}\right) = \left(1 - \frac{2}{x}\right)x^2$$
$$2 = x^2 - 2x$$

Make one side zero.

$$0 = x^2 - 2x - 2$$

The right-hand side is a quadratic trinomial with $ac = (1)(-2) = -2$. There are no integer pairs with product -2 that sum to -2, so this quadratic trinomial does not factor. Fortunately, the equation is quadratic (second degree), so we can use the quadratic formula with $a = 1$, $b = -2$, and $c = -2$.

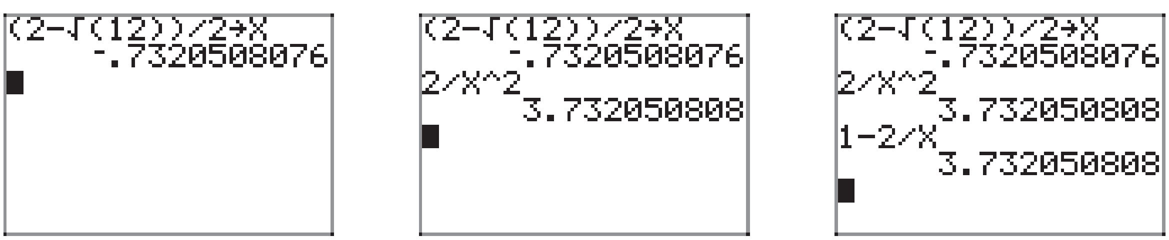
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)} = \frac{2 \pm \sqrt{12}}{2}$$

This gives us two solutions, $x = (2 - \sqrt{12})/2$ and $x = (2 + \sqrt{12})/2$. Let's check the solution $x = (2 - \sqrt{12})/2$. First, enter this result in your calculator, press the `STO▸` button, press `X`, then press the `ENTER` key to execute the command and store the solution in the variable `X`. This command is shown in **Figure 2**(a).

Enter the left-hand side of the original **equation (6)** as `2/x^2` and press the `ENTER` key to execute this command. This is shown in **Figure 2**(b).

Enter the right-hand side of the original **equation (6)** as `1-2/X` and press the `ENTER` key to execute this command. This is shown in **Figure 2**(c). Note that the left- and right-hand sides of **equation (6)** are both shown to equal `3.732050808` at $x = (2 - \sqrt{12})/2$ (at `X = -0.7320508076`), as shown in **Figure 2**(c). This shows that $x = (2 - \sqrt{12})/2$ is a solution of **equation (6)**.

We leave it to our readers to check the second solution, $x = (2 + \sqrt{12})/2$.



(a) (b) (c)

Figure 2. Using the graphing calculator to check the solutions of **equation (6)**.

Let's look at another example, this one involving function notation.

Example 7. *Consider the function defined by*

$$f(x) = \frac{1}{x} + \frac{1}{x-4}. \tag{8}$$

Solve the equation $f(x) = 2$ for x using both graphical and analytical techniques, then compare solutions. Perform each of the following tasks.

a. *Sketch the graph of f on graph paper. Label the zeros of f with their coordinates and the asymptotes of f with their equations.*
b. *Add the graph of $y = 2$ to your plot and estimate the coordinates of where the graph of f intersects the graph of $y = 2$.*
c. *Use the `intersect` utility on your calculator to find better approximations of the points where the graphs of f and $y = 2$ intersect.*
d. *Solve the equation $f(x) = 2$ algebraically and compare your solutions to those found in part (c).*

For the graph in part (a), we need to find the zeros of f and the equations of any vertical or horizontal asymptotes.

To find the zero of the function f, we find a common denominator and add the two rational expressions in **equation (8)**.

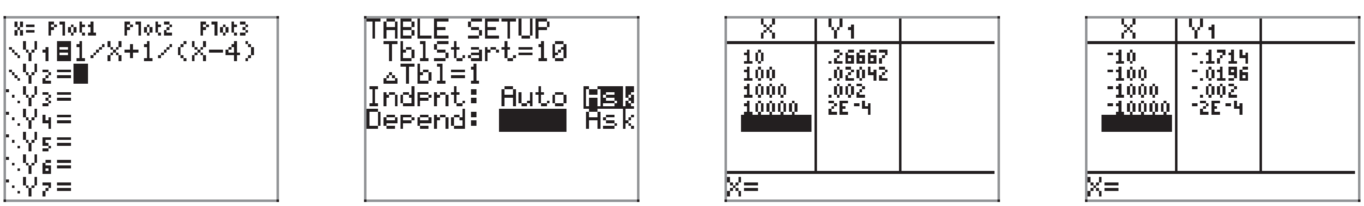
$$f(x) = \frac{1}{x} + \frac{1}{x-4} = \frac{x-4}{x(x-4)} + \frac{x}{x(x-4)} = \frac{2x-4}{x(x-4)} \tag{9}$$

Note that the numerator of this result equal zero (but not the denominator) when $x = 2$. This is the zero of f. Thus, the graph of f has x-intercept at $(2, 0)$, as shown in **Figure 4**.

Note that the rational function in **equation (9)** is reduced to lowest terms. The denominators of x and x - 4 in **equation (9)** are zero when $x = 0$ and $x = 4$. These are our vertical asymptotes, as shown in **Figure 4**.

To find the horizontal asymptotes, we need to examine what happens to the function values as x increases (or decreases) without bound. Enter the function in the `Y=` menu with `1/X+1/(X-4)`, as shown in **Figure 3**(a). Press `2nd TBLSET`, then highlight `ASK` for the independent variable and press `ENTER` to make this selection permanent, as shown in **Figure 3**(b).

Press `2nd TABLE`, then enter 10, 100, 1,000, and 10,000, as shown in **Figure 3**(c). Note how the values of `Y1` approach zero. In **Figure 3**(d), as x decreases without bound, the end-behavior is the same. This is an indication of a horizontal asymptote at $y = 0$, as shown in **Figure 4**.



(a) (b) (c) (d)

Figure 3. Examining the end-behavior of f with the graphing calculator.

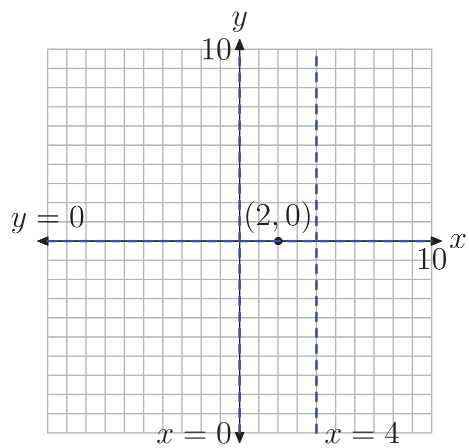


Figure 4. Placing the horizontal and vertical asymptotes and the x-intercept of the graph of the function f.

At this point, we already have our function f loaded in `Y1`, so we can press the `ZOOM` button and select `6:ZStandard` to produce the graph shown in **Figure 5**. As expected, the graphing calculator does not do a very good job with the rational function f, particularly near the discontinuities at the vertical asymptotes. However, there is enough information in **Figure 5**, couple with our advanced work summarized in **Figure 4**, to draw a very nice graph of the rational function on our graph paper, as shown in **Figure 6**(a). *Note: We haven't labeled asymptotes with equations, nor zeros with coordinates, in **Figure 6**(a), as we thought the picture might be a little crowded. However, you should label each of these parts on your graph paper, as we did in **Figure 4**.*

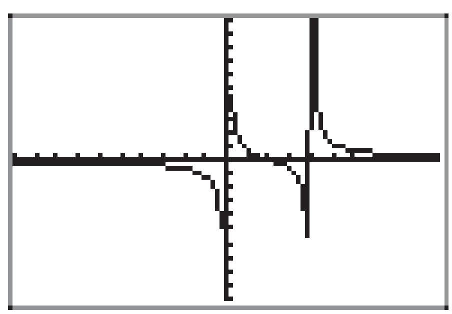


Figure 5. The graph of f as drawn on the calculator.

Let's now address part (b) by adding the horizontal line $y = 2$ to the graph, as shown in **Figure 6**(b). Note that the graph of $y = 2$ intersects the graph of the rational function f at two points A and B. The x-values of points A and B are the solutions to our equation $f(x) = 2$.

We can get a crude estimate of the x-coordinates of points A and B right off our graph paper. The x-value of point A is approximately $x \approx 0.3$, while the x-value of point B appears to be approximately $x \approx 4.6$.

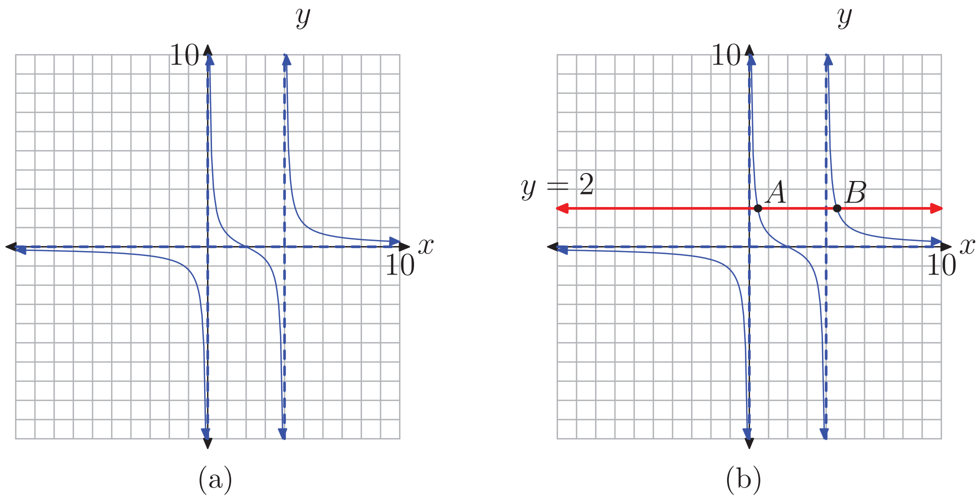


Figure 6. Solving $f(x) = 2$ graphically.

Next, let's address the task required in part (c). We have very reasonable estimates of the solutions of $f(x) = 2$ based on the data presented in **Figure 6**(b). Let's use the graphing calculator to improve upon these estimates.

First, load the equation `Y2=2` into the `Y=` menu, as shown in **Figure 7**(a). We need to find where the graph of `Y1` intersects the graph of `Y2`, so we press `2nd CALC` and select `5:intersect` from the menu. In the usual manner, select "First curve," "Second curve," and move the cursor close to the point you wish to estimate. This is your "Guess." Perform similar tasks for the second point of intersection.

Our results are shown in **Figures 7**(b) and **Figures 7**(c). The estimate in **Figure 7**(b) has $x \approx 0.43844719$, while that in **Figure 7**(c) has $x \approx 4.5615528$. Note that these are more accurate than the approximations of $x \approx 0.3$ and $x \approx 4.6$ captured from our hand drawn image in **Figure 6**(b).

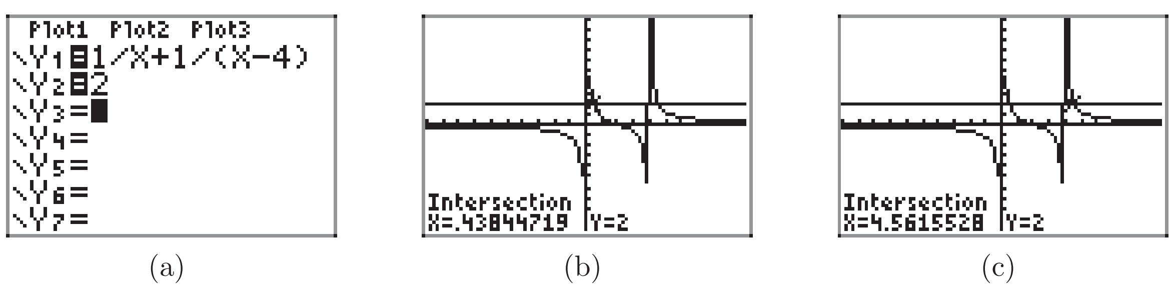


Figure 7. Solving $f(x) = 2$ graphically.

Finally, let's address the request for an algebraic solution of $f(x) = 2$ in part (d). First, replace $f(x)$ with $1/x + 1/(x-4)$ to obtain

$$f(x) = 2$$
$$\frac{1}{x} + \frac{1}{x-4} = 2.$$

Multiply both sides of this equation by the common denominator $x(x-4)$.

$$\begin{aligned} x(x-4)\left[\frac{1}{x}+\frac{1}{x-4}\right] &= [2]\,x(x-4) \\ x(x-4)\left[\frac{1}{x}\right]+x(x-4)\left[\frac{1}{x-4}\right] &= [2]\,x(x-4) \end{aligned}$$

Cancel.

$$\begin{aligned} \cancel{x}(x-4)\left[\frac{1}{\cancel{x}}\right]+x\cancel{(x-4)}\left[\frac{1}{\cancel{x-4}}\right] &= [2]\,x(x-4) \\ (x-4)+x &= 2x(x-4) \end{aligned}$$

Simplify each side.

$$2x-4=2x^2-8x$$

This last equation is nonlinear, so we make one side zero by subtracting $2x$ and adding 4 to both sides of the equation.

$$\begin{aligned} 0 &= 2x^2-8x-2x+4 \\ 0 &= 2x^2-10x+4 \end{aligned}$$

Note that each coefficient on the right-hand side of this last equation is divisible by 2. Let's divide both sides of the equation by 2, distributing the division through each term on the right-hand side of the equation.

$$0=x^2-5x+2$$

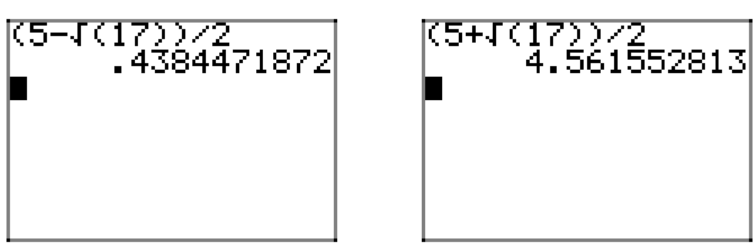
The trinomial on the right is a quadratic with $ac=(1)(2)=2$. There are no integer pairs having product 2 and sum -5, so this trinomial doesn't factor. We will use the quadratic formula instead, with $a=1$, $b=-5$ and $c=2$.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(2)}}{2(1)}=\frac{5\pm\sqrt{17}}{2}$$

It remains to compare these with the graphical solutions found in part (c). So, enter the solution `(5-√(17))/(2)` in your calculator screen, as shown in **Figure 8**(a). Enter `(5+√(17))/(2)`, as shown in **Figure 8**(b). Thus,

$$\frac{5-\sqrt{17}}{2}\approx 0.4384471872 \qquad \text{and} \qquad \frac{5+\sqrt{17}}{2}\approx 4.561552813.$$

Note the close agreement with the approximations found in part (c).



(a) (b)

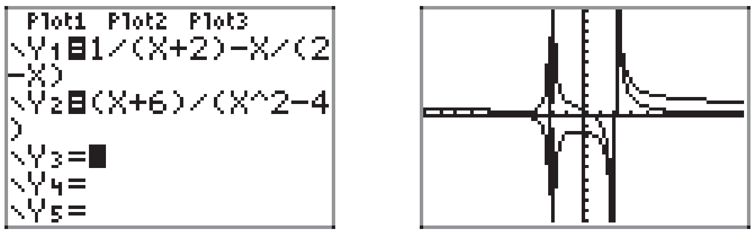
Figure 8. Approximating the exact solutions.

Let's look at another example.

Example 10. *Solve the following equation for x, both graphically and analytically.*

$$\frac{1}{x+2} - \frac{x}{2-x} = \frac{x+6}{x^2-4} \tag{11}$$

We start the graphical solution in the usual manner, loading the left- and right-hand sides of **equation (11)** into `Y1` and `Y2`, as shown in **Figure 9**(a). Note that in the resulting plot, shown in **Figure 9**(b), it is very difficult to interpret where the graph of the left-hand side intersects the graph of the right-hand side of **equation (11)**.



(a) (b)

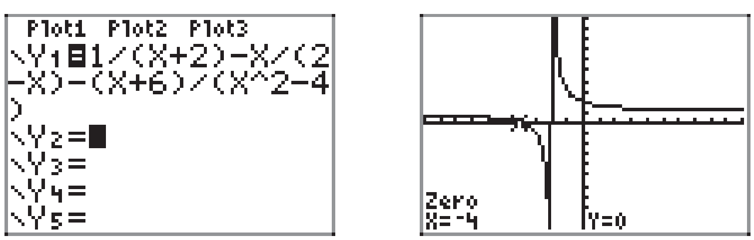
Figure 9. Sketch the left- and right-hand sides of **equation (11)**.

In this situation, a better strategy is to make one side of **equation (11)** equal to zero.

$$\frac{1}{x+2} - \frac{x}{2-x} - \frac{x+6}{x^2-4} = 0 \tag{12}$$

Our approach will now change. We'll plot the left-hand side of **equation (12)**, then find where the left-hand side is equal to zero; that is, we'll find where the graph of the left-hand side of **equation (12)** intercepts the x-axis.

With this thought in mind, load the left-hand side of **equation (12)** into `Y1`, as shown in **Figure 10**(a). Note that the graph in **Figure 10**(b) appears to have only one vertical asymptote at $x = -2$ (some cancellation must remove the factor of $x - 2$ from the denominator when you combine the terms of the left-hand side of **equation (12)**[20]). Further, when you use the `zero` utility in the `CALC` menu of the graphing calculator, there appears to be a zero at $x = -4$, as shown in **Figure 10**(b).



(a) (b)

Figure 10. Finding the zero of the left-hand side of **equation (12)**.

[20] Closer analysis might reveal a "hole" in the graph, but we push on because our check at the end of the problem will reveal a false solution.

Therefore, **equation (12)** seems to have only one solution, namely $x = 4$.

Next, let's seek an analytical solution of **equation (11)**. We'll need to factor the denominators in order to discover a common denominator.

$$\frac{1}{x+2} - \frac{x}{2-x} = \frac{x+6}{(x+2)(x-2)}$$

It's tempting to use a denominator of $(x+2)(2-x)(x-2)$. However, the denominator of the second term on the left-hand side of this last equation, $2-x$, is in a different order than the factors in the other denominators, $x-2$ and $x+2$, so let's perform a sign change on this term and reverse the order. We will negate the fraction bar and negate the denominator. That's two sign changes, so the term remains unchanged when we write

$$\frac{1}{x+2} + \frac{x}{x-2} = \frac{x+6}{(x+2)(x-2)}.$$

Now we see that a common denominator of $(x+2)(x-2)$ will suffice. Let's multiply both sides of the last equation by $(x+2)(x-2)$.

$$(x+2)(x-2)\left[\frac{1}{x+2} + \frac{x}{x-2}\right] = \left[\frac{x+6}{(x+2)(x-2)}\right](x+2)(x-2)$$

$$(x+2)(x-2)\left[\frac{1}{x+2}\right] + (x+2)(x-2)\left[\frac{x}{x-2}\right] = \left[\frac{x+6}{(x+2)(x-2)}\right](x+2)(x-2)$$

Cancel.

$$\cancel{(x+2)}(x-2)\left[\frac{1}{\cancel{x+2}}\right] + (x+2)\cancel{(x-2)}\left[\frac{x}{\cancel{x-2}}\right] = \left[\frac{x+6}{\cancel{(x+2)}\cancel{(x-2)}}\right]\cancel{(x+2)}\cancel{(x-2)}$$

$$(x-2) + x(x+2) = x+6$$

Simplify.

$$x - 2 + x^2 + 2x = x + 6$$
$$x^2 + 3x - 2 = x + 6$$

This last equation is nonlinear because of the presence of a power of x larger than 1 (note the x^2 term). Therefore, the strategy is to make one side of the equation equal to zero. We will subtract x and subtract 6 from both sides of the equation.

$$x^2 + 3x - 2 - x - 6 = 0$$
$$x^2 + 2x - 8 = 0$$

The left-hand side is a quadratic trinomial with $ac = (1)(-8) = -8$. The integer pair 4 and -2 have product -8 and sum 2. Thus,

$$(x+4)(x-2) = 0.$$

Using the zero product property, either

$$x + 4 = 0 \qquad \text{or} \qquad x - 2 = 0,$$

so

$$x = -4 \qquad \text{or} \qquad x = 2.$$

The fact that we have found two answers using an analytical method is troubling. After all, the graph in **Figure 10**(b) indicates only one solution, namely $x = -4$. It is comforting that one of our analytical solutions is also $x = -4$, but it is still disconcerting that our analytical approach reveals a second "answer," namely $x = 2$.

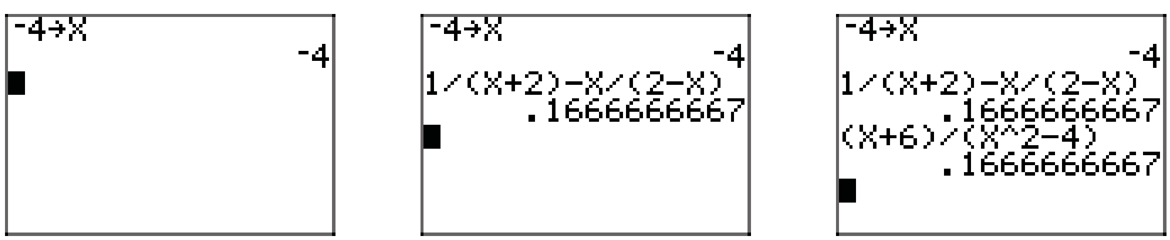
However, notice that we haven't paid any attention to the restrictions caused by denominators up to this point. Indeed, careful consideration of **equation (11)** reveals factors of $x+2$ and $x-2$ in the denominators. Hence, $x = -2$ and $x = 2$ are restrictions.

Note that one of our answers, namely $x = 2$, is a restricted value. It will make some of the denominators in **equation (11)** equal to zero, so it cannot be a solution. Thus, the only viable solution is $x = -4$. One can certainly check this solution by hand, but let's use the graphing calculator to assist us in the check.

First, enter `-4`, press the `STOI` button, press `X`, then press `ENTER` to execute the resulting command and store `-4` in the variable `X`. The result is shown in **Figure 11**(a).

Next, we calculate the value of the left-hand side of **equation (11)** at this value of `X`. Enter the left-hand side of **equation (11)** as `1/(X+2)-X/(2-X)`, then press the `ENTER` key to execute the statement and produce the result shown in **Figure 11**(b).

Finally, enter the right-hand side of **equation (11)** as `(X+6)/(x^2-4)` and press the `ENTER` key to execute the statement. The result is shown in **Figure 11**(c). Note that both sides of the equation equal `.1666666667` at `X=-4`. Thus, the solution $x = -4$ checks.



(a) (b) (c)

Figure 11. Using the graphing calculator to check the solution $x = -4$ of **equation (11)**.

3.6 *Exercises*

For each of the rational functions given in **Exercises 1-6**, perform each of the following tasks.

i. Set up a coordinate system on graph paper. Label and scale each axis. *Remember to draw all lines with a ruler.*
ii. Plot the zero of the rational function on your coordinate system and label it with its coordinates. Plot the vertical and horizontal asymptotes on your coordinate system and label them with their equations. Use this information (and your graphing calculator) to draw the graph of f.
iii. Plot the horizontal line $y = k$ on your coordinate system and label this line with its equation.
iv. Use your calculator's `intersect` utility to help determine the solution of $f(x) = k$. Label this point on your graph with its coordinates.
v. Solve the equation $f(x) = k$ algebraically, placing the work for this solution on your graph paper next to your coordinate system containing the graphical solution. Do the answers agree?

1. $f(x) = \dfrac{x-1}{x+2}; \quad k = 3$

2. $f(x) = \dfrac{x+1}{x-2}; \quad k = -3$

3. $f(x) = \dfrac{x+1}{3-x}; \quad k = 2$

4. $f(x) = \dfrac{x+3}{2-x}; \quad k = 2$

5. $f(x) = \dfrac{2x+3}{x-1}; \quad k = -3$

6. $f(x) = \dfrac{5-2x}{x-1}; \quad k = 3$

In **Exercises 7-14**, use a strictly algebraic technique to solve the equation $f(x) = k$ for the given function and value of k. You are encouraged to check your result with your calculator.

7. $f(x) = \dfrac{16x-9}{2x-1}; \quad k = 8$

8. $f(x) = \dfrac{10x-3}{7x+7}; \quad k = 1$

9. $f(x) = \dfrac{5x+8}{4x+1}; \quad k = -11$

10. $f(x) = -\dfrac{6x-11}{7x-2}; \quad k = -6$

11. $f(x) = -\dfrac{35x}{7x+12}; \quad k = -5$

12. $f(x) = -\dfrac{66x-5}{6x-10}; \quad k = -11$

13. $f(x) = \dfrac{8x+2}{x-11}; \quad k = 11$

14. $f(x) = \dfrac{36x-7}{3x-4}; \quad k = 12$

In **Exercises 15-20**, use a strictly algebraic technique to solve the given equation. You are encouraged to check your result with your calculator.

15. $\dfrac{x}{7} + \dfrac{8}{9} = -\dfrac{8}{7}$

16. $\dfrac{x}{3} + \dfrac{9}{2} = -\dfrac{3}{8}$

17. $-\dfrac{57}{x} = 27 - \dfrac{40}{x^2}$

18. $-\dfrac{117}{x} = 54 + \dfrac{54}{x^2}$

19. $\dfrac{7}{x} = 4 - \dfrac{3}{x^2}$

20. $\dfrac{3}{x^2} = 5 - \dfrac{3}{x}$

For each of the rational functions given in **Exercises 21-26**, perform each of the following tasks.

i. Set up a coordinate system on graph paper. Label and scale each axis. *Remember to draw all lines with a ruler.*
ii. Plot the zero of the rational function on your coordinate system and label it with its coordinates. You may use your calculator's `zero` utility to find this, if you wish.
iii. Plot the vertical and horizontal asymptotes on your coordinate system and label them with their equations. Use the asymptote and zero information (and your graphing calculator) to draw the graph of f.
iv. Plot the horizontal line $y = k$ on your coordinate system and label this line with its equation.
v. Use your calculator's `intersect` utility to help determine the solution of $f(x) = k$. Label this point on your graph with its coordinates.
vi. Solve the equation $f(x) = k$ algebraically, placing the work for this soluton on your graph paper next to your coordinate system containing the graphical solution. Do the answers agree?

21. $f(x) = \dfrac{1}{x} + \dfrac{1}{x+5}, \quad k = 9/14$

22. $f(x) = \dfrac{1}{x} + \dfrac{1}{x-2}, \quad k = 8/15$

23. $f(x) = \dfrac{1}{x-1} - \dfrac{1}{x+1}, \quad k = 1/4$

24. $f(x) = \dfrac{1}{x-1} - \dfrac{1}{x+2}, \quad k = 1/6$

25. $f(x) = \dfrac{1}{x-2} + \dfrac{1}{x+2}, \quad k = 4$

26. $f(x) = \dfrac{1}{x-3} + \dfrac{1}{x+2}, \quad k = 5$

In **Exercises 27-34**, use a strictly algebraic technique to solve the given equation. You are encouraged to check your result with your calculator.

27. $\dfrac{2}{x+1} + \dfrac{4}{x+2} = -3$

28. $\dfrac{2}{x-5} - \dfrac{7}{x-7} = 9$

29. $\dfrac{3}{x+9} - \dfrac{2}{x+7} = -3$

30. $\dfrac{3}{x+9} - \dfrac{6}{x+7} = 9$

31. $\dfrac{2}{x+9} + \dfrac{2}{x+6} = -1$

32. $\dfrac{5}{x-6} - \dfrac{8}{x-7} = -1$

33. $\dfrac{3}{x+3} + \dfrac{6}{x+2} = -2$

34. $\dfrac{2}{x-4} - \dfrac{2}{x-1} = 1$

For each of the equations in **Exercises 35-40**, perform each of the following tasks.

i. Follow the lead of Example 10 in the text. Make one side of the equation equal to zero. Load the nonzero side into your calculator and draw its graph.
ii. Determine the vertical asymptotes of equation and the re-sulting graph on your calculator. Use the `TABLE` feature of your calculator to determine any horizontal asymptote behavior.
iii. Use the `zero` finding utility in the `CALC` menu to determine the zero of the nonzero side of the resulting equation.
iv. Set up a coordinate system on graph paper. Label and scale each axis. *Remember to draw all lines with a ruler.* Draw the graph of the nonzero side of the equation. Draw the vertical and horizontal asymptotes and label them with their equations. Plot the x-intercept and label it with its coordinates.
v. Use an algebraic technique to determine the solution of the equation and compare it with the solution found by the graphical analysis above.

35. $\dfrac{x}{x+1}+\dfrac{8}{x^2-2x-3}=\dfrac{2}{x-3}$

36. $\dfrac{x}{x+4}-\dfrac{2}{x+1}=\dfrac{12}{x^2+5x+4}$

37. $\dfrac{x}{x+1}-\dfrac{4}{2x+1}=\dfrac{2x-1}{2x^2+3x+1}$

38. $\dfrac{2x}{x-4}-\dfrac{1}{x+1}=\dfrac{4x+24}{x^2-3x-4}$

39. $\dfrac{x}{x-2}+\dfrac{3}{x+2}=\dfrac{8}{4-x^2}$

40. $\dfrac{x}{x-1}-\dfrac{4}{x+1}=\dfrac{x-6}{1-x^2}$

In **Exercises 41-68**, use a strictly algebraic technique to solve the given equation. You are encouraged to check your result with your calculator.

41. $\dfrac{x}{3x-9}-\dfrac{9}{x}=\dfrac{1}{x-3}$

42. $\dfrac{5x}{x+2}+\dfrac{5}{x-5}=\dfrac{x+6}{x^2-3x-10}$

43. $\dfrac{3x}{x+2}-\dfrac{7}{x}=-\dfrac{1}{2x+4}$

44. $\dfrac{4x}{x+6}-\dfrac{4}{x+4}=\dfrac{x-4}{x^2+10x+24}$

45. $\dfrac{x}{x-5}+\dfrac{9}{4-x}=\dfrac{x+5}{x^2-9x+20}$

46. $\dfrac{6x}{x-5}-\dfrac{2}{x-3}=\dfrac{x-8}{x^2-8x+15}$

47. $\dfrac{2x}{x-4}+\dfrac{5}{2-x}=\dfrac{x+8}{x^2-6x+8}$

48. $\dfrac{x}{x-7}-\dfrac{8}{5-x}=\dfrac{x+7}{x^2-12x+35}$

49. $-\dfrac{x}{2x+2}-\dfrac{6}{x}=-\dfrac{2}{x+1}$

50. $\dfrac{7x}{x+3}-\dfrac{4}{2-x}=\dfrac{x+8}{x^2+x-6}$

51. $\dfrac{2x}{x+5}-\dfrac{2}{6-x}=\dfrac{x-2}{x^2-x-30}$

52. $\dfrac{4x}{x+1}+\dfrac{6}{x+3}=\dfrac{x-9}{x^2+4x+3}$

53. $\dfrac{x}{x+7}-\dfrac{2}{x+5}=\dfrac{x+1}{x^2+12x+35}$

54. $\dfrac{5x}{6x+4}+\dfrac{6}{x}=\dfrac{1}{3x+2}$

55. $\dfrac{2x}{3x+9}-\dfrac{4}{x}=-\dfrac{2}{x+3}$

56. $\dfrac{7x}{x+1} - \dfrac{4}{x+2} = \dfrac{x+6}{x^2+3x+2}$

57. $\dfrac{x}{2x-8} + \dfrac{8}{x} = \dfrac{2}{x-4}$

58. $\dfrac{3x}{x-6} + \dfrac{6}{x-6} = \dfrac{x+2}{x^2-12x+36}$

59. $\dfrac{x}{x+2} + \dfrac{2}{x} = -\dfrac{5}{2x+4}$

60. $\dfrac{4x}{x-2} + \dfrac{2}{2-x} = \dfrac{x+4}{x^2-4x+4}$

61. $-\dfrac{2x}{3x-9} - \dfrac{3}{x} = -\dfrac{2}{x-3}$

62. $\dfrac{2x}{x+1} - \dfrac{2}{x} = \dfrac{1}{2x+2}$

63. $\dfrac{x}{x+1} + \dfrac{5}{x} = \dfrac{1}{4x+4}$

64. $\dfrac{2x}{x-4} - \dfrac{8}{x-7} = \dfrac{x+2}{x^2-11x+28}$

65. $-\dfrac{9x}{8x-2} + \dfrac{2}{x} = -\dfrac{2}{4x-1}$

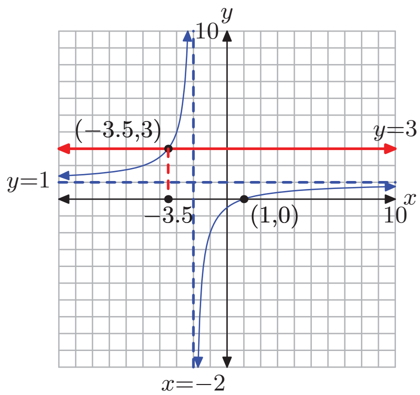
66. $\dfrac{2x}{x-3} - \dfrac{4}{4-x} = \dfrac{x-9}{x^2-7x+12}$

67. $\dfrac{4x}{x+6} - \dfrac{5}{7-x} = \dfrac{x-5}{x^2-x-42}$

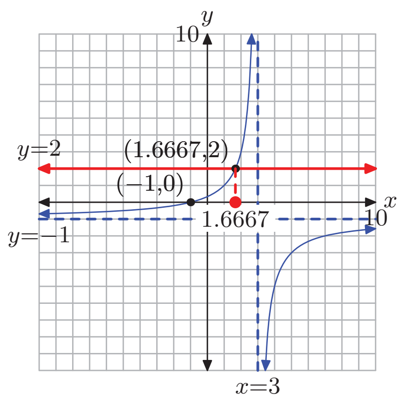
68. $\dfrac{x}{x-1} - \dfrac{4}{x} = \dfrac{1}{5x-5}$

3.6 *Answers*

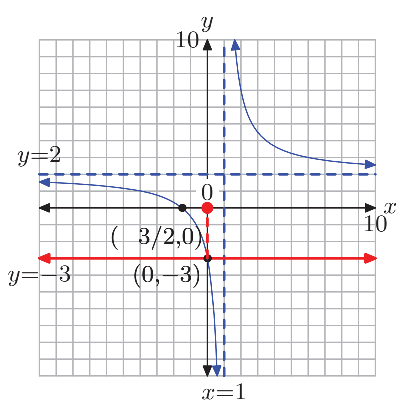
1. $x = -7/2$



3. $x = 5/3$



5. $x = 0$



7. none

9. $-\dfrac{19}{49}$

11. none

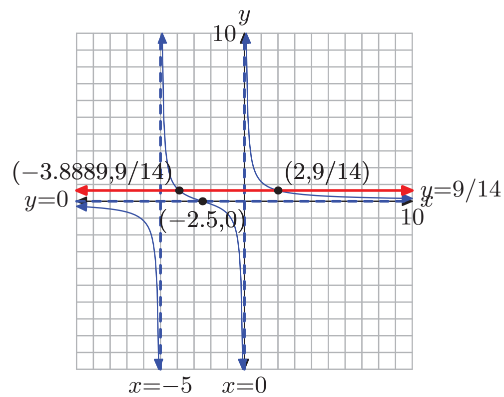
13. 41

15. $-\dfrac{128}{9}$

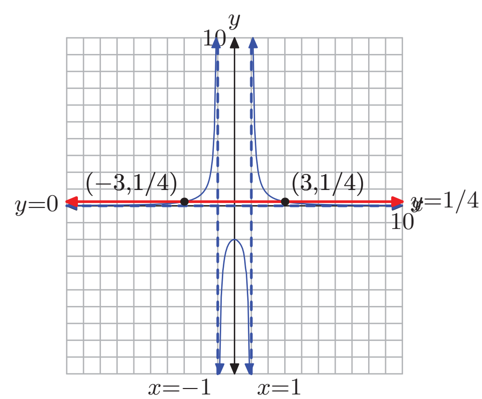
17. $-\dfrac{8}{3}, \dfrac{5}{9}$

19. $\dfrac{7+\sqrt{97}}{8}, \dfrac{7-\sqrt{97}}{8}$

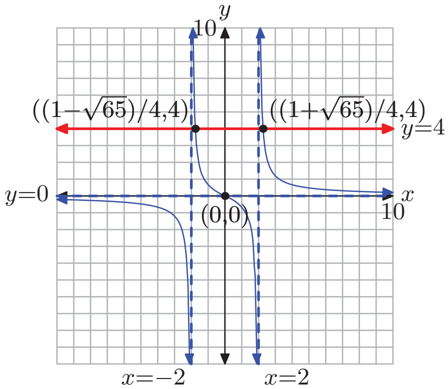
21. $x = -35/9$ or $x = 2$



23. $x = -3$ or $x = 3$



25. $x = \dfrac{1+\sqrt{65}}{4}, \dfrac{1-\sqrt{65}}{4}$



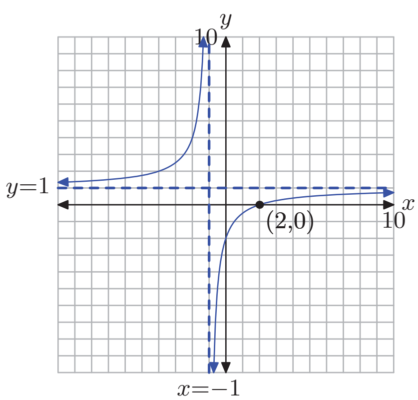
27. $\dfrac{-15+\sqrt{57}}{6}, \dfrac{-15-\sqrt{57}}{6}$

29. $\dfrac{-49+\sqrt{97}}{6}, \dfrac{-49-\sqrt{97}}{6}$

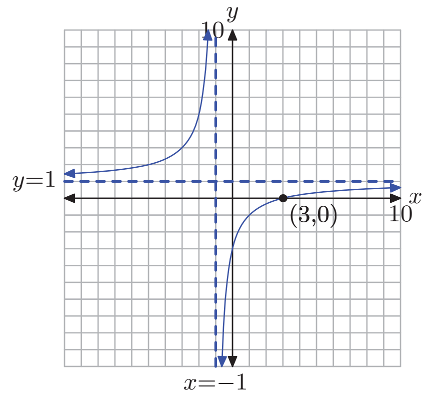
31. $-7, -12$

33. $\dfrac{-19+\sqrt{73}}{4}, \dfrac{-19-\sqrt{73}}{4}$

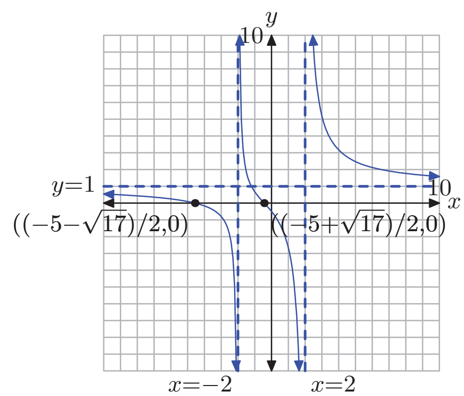
35. $x = 2$



37. $x = 3$



39. $x = \dfrac{-5+\sqrt{17}}{2}, \dfrac{-5-\sqrt{17}}{2}$



41. 27

43. $\dfrac{7}{2}, -\dfrac{4}{3}$

45. 10

47. 3

49. $-6, -2$

51. $4, \frac{3}{2}$

53. 3

55. 6

57. -16

59. $\dfrac{-9+\sqrt{17}}{4}, \dfrac{-9-\sqrt{17}}{4}$

61. $-\dfrac{9}{2}$

63. $\dfrac{-19+\sqrt{41}}{8}, \dfrac{-19-\sqrt{41}}{8}$

65. $\dfrac{2}{9}, 2$

67. $\frac{7}{2}, \frac{5}{2}$

3.7 Applications of Rational Functions

In this section, we will investigate the use of rational functions in several applications.

Number Problems

We start by recalling the definition of the *reciprocal* of a number.

Definition 1. *For any nonzero real number a, the* **reciprocal** *of a is the number $1/a$. Note that the product of a number and its reciprocal is always equal to the number 1. That is,*

$$a \cdot \frac{1}{a} = 1.$$

For example, the reciprocal of the number 3 is 1/3. Note that we simply "invert" the number 3 to obtain its reciprocal 1/3. Further, note that the product of 3 and its reciprocal 1/3 is

$$3 \cdot \frac{1}{3} = 1.$$

As a second example, to find the reciprocal of $-3/5$, we could make the calculation

$$\frac{1}{-\frac{3}{5}} = 1 \div \left(-\frac{3}{5}\right) = 1 \cdot \left(-\frac{5}{3}\right) = -\frac{5}{3},$$

but it's probably faster to simply "invert" $-3/5$ to obtain its reciprocal $-5/3$. Again, note that the product of $-3/5$ and its reciprocal $-5/3$ is

$$\left(-\frac{3}{5}\right) \cdot \left(-\frac{5}{3}\right) = 1.$$

Let's look at some applications that involve the reciprocals of numbers.

I **Example 2.** *The sum of a number and its reciprocal is* 29/10*. Find the number(s).*

Let x represent a nonzero number. The reciprocal of x is $1/x$. Hence, the sum of x and its reciprocal is represented by the rational expression $x + 1/x$. Set this equal to 29/10.

$$x + \frac{1}{x} = \frac{29}{10}$$

To clear fractions from this equation, multiply both sides by the common denominator $10x$.

$$10x\left(x+\frac{1}{x}\right)=\left(\frac{29}{10}\right)10x$$
$$10x^2+10=29x$$

This equation is nonlinear (it has a power of x larger than 1), so make one side equal to zero by subtracting $29x$ from both sides of the equation.

$$10x^2-29x+10=0$$

Let's try to use the ac-test to factor. Note that $ac=(10)(10)=100$. The integer pair $\{-4,-25\}$ has product 100 and sum -29. Break up the middle term of the quadratic trinomial using this pair, then factor by grouping.

$$\begin{aligned} 10x^2-4x-25x+10&=0\\ 2x(5x-2)-5(5x-2)&=0\\ (2x-5)(5x-2)&=0 \end{aligned}$$

Using the zero product property, either

$$2x-5=0 \qquad \text{or} \qquad 5x-2=0.$$

Each of these linear equations is easily solved.

$$x=\frac{5}{2} \qquad \text{or} \qquad x=\frac{2}{5}$$

Hence, we have two solutions for x. However, they both lead to the same number-reciprocal pair. That is, if $x=5/2$, then its reciprocal is $2/5$. On the other hand, if $x=2/5$, then its reciprocal is $5/2$.

Let's check our solution by taking the sum of the solution and its reciprocal. Note that

$$\frac{5}{2}+\frac{2}{5}=\frac{25}{10}+\frac{4}{10}=\frac{29}{10},$$

as required by the problem statement.

Let's look at another application of the reciprocal concept.

I **Example 3.** *There are two numbers. The second number is 1 larger than twice the first number. The sum of the reciprocals of the two numbers is $7/10$. Find the two numbers.*

Let x represent the first number. If the second number is 1 larger than twice the first number, then the second number can be represented by the expression $2x+1$.

Thus, our two numbers are x and $2x+1$. Their reciprocals, respectively, are $1/x$ and $1/(2x+1)$. Therefore, the sum of their reciprocals can be represented by the rational expression $1/x+1/(2x+1)$. Set this equal to $7/10$.

$$\frac{1}{x} + \frac{1}{2x+1} = \frac{7}{10}$$

Multiply both sides of this equation by the common denominator $10x(2x+1)$.

$$10x(2x+1)\left[\frac{1}{x} + \frac{1}{2x+1}\right] = \left[\frac{7}{10}\right]10x(2x+1)$$
$$10(2x+1) + 10x = 7x(2x+1)$$

Expand and simplify each side of this result.

$$\begin{aligned} 20x + 10 + 10x &= 14x^2 + 7x \\ 30x + 10 &= 14x^2 + 7x \end{aligned}$$

Again, this equation is nonlinear. We will move everything to the right-hand side of this equation. Subtract $30x$ and 10 from both sides of the equation to obtain

$$\begin{aligned} 0 &= 14x^2 + 7x - 30x - 10 \\ 0 &= 14x^2 - 23x - 10. \end{aligned}$$

Note that the right-hand side of this equation is quadratic with $ac = (14)(-10) = -140$. The integer pair $\{5, -28\}$ has product -140 and sum -23. Break up the middle term using this pair and factor by grouping.

$$\begin{aligned} 0 &= 14x^2 + 5x - 28x - 10 \\ 0 &= x(14x+5) - 2(14x+5) \\ 0 &= (x-2)(14x+5) \end{aligned}$$

Using the zero product property, either

$$x - 2 = 0 \qquad \text{or} \qquad 14x + 5 = 0.$$

These linear equations are easily solved for x, providing

$$x = 2 \qquad \text{or} \qquad x = -\frac{5}{14}.$$

We still need to answer the question, which was to find two numbers such that the sum of their reciprocals is $7/10$. Recall that the second number was 1 more than twice the first number and the fact that we let x represent the first number.

Consequently, if the first number is $x = 2$, then the second number is $2x + 1$, or $2(2) + 1$. That is, the second number is 5. Let's check to see if the pair $\{2, 5\}$ is a solution by computing the sum of the reciprocals of 2 and 5.

$$\frac{1}{2} + \frac{1}{5} = \frac{5}{10} + \frac{2}{10} = \frac{7}{10}$$

Thus, the pair $\{2, 5\}$ is a solution.

However, we found a second value for the first number, namely $x = -5/14$. If this is the first number, then the second number is

$$2\left(-\frac{5}{14}\right) + 1 = -\frac{5}{7} + \frac{7}{7} = \frac{2}{7}.$$

Thus, we have a second pair $\{-5/14, 2/7\}$, but what is the sum of the reciprocals of these two numbers? The reciprocals are $-14/5$ and $7/2$, and their sum is

$$-\frac{14}{5} + \frac{7}{2} = -\frac{28}{10} + \frac{35}{10} = \frac{7}{10},$$

as required by the problem statement. Hence, the pair $\{-14/5, 7/2\}$ is also a solution.

Distance, Speed, and Time Problems

When we developed the *Equations of Motion* in the chapter on quadratic functions, we showed that if an object moves with constant speed, then the distance traveled is given by the formula

$$d = vt, \tag{4}$$

where d represents the distance traveled, v represents the speed, and t represents the time of travel.

For example, if a car travels down a highway at a constant speed of 50 miles per hour (50 mi/h) for 4 hours (4 h), then it will travel

$$\begin{aligned} d &= vt \\ d &= 50\,\frac{\text{mi}}{\text{h}} \times 4\,\text{h} \\ d &= 200\,\text{mi}. \end{aligned}$$

Let's put this relation to use in some applications.

I **Example 5.** *A boat travels at a constant speed of 3 miles per hour in still water. In a river with unknown current, it takes the boat twice as long to travel 60 miles upstream (against the current) than it takes for the 60 mile return trip (with the current). What is the speed of the current in the river?*

The speed of the boat in still water is 3 miles per hour. When the boat travels upstream, the current is against the direction the boat is traveling and works to reduce the actual speed of the boat. When the boat travels downstream, then the actual speed of the boat is its speed in still water increased by the speed of the current. If we let c represent the speed of the current in the river, then the boat's speed upstream (against the current) is $3 - c$, while the boat's speed downstream (with the current) is $3 + c$. Let's summarize what we know in a distance-speed-time table (see **Table 1**).

	d (mi)	v (mi/h)	t (h)
Upstream	60	$3-c$	?
Downstream	60	$3+c$	?

Table 1. A distance, speed, and time table.

Here is a useful piece of advice regarding distance, speed, and time tables.

Distance, Speed, and Time Tables. Because distance, speed, and time are related by the equation $d = vt$, whenever you have two boxes in a row of the table completed, the third box in that row can be calculated by means of the formula $d = vt$.

Note that each row of **Table 1** has two entries entered. The third entry in each row is time. Solve the equation $d = vt$ for t to obtain

$$t = \frac{d}{v}.$$

The relation $t = d/v$ can be used to compute the time entry in each row of **Table 1**.

For example, in the first row, $d = 60$ miles and $v = 3-c$ miles per hour. Therefore, the time of travel is

$$t = \frac{d}{v} = \frac{60}{3-c}.$$

Note how we've filled in this entry in **Table 2**. In similar fashion, the time to travel downstream is calculated with

$$t = \frac{d}{v} = \frac{60}{3+c}.$$

We've also added this entry to the time column in **Table 2**.

	d (mi)	v (mi/h)	t (h)
Upstream	60	$3-c$	$\dfrac{60}{3-c}$
Downstream	60	$3+c$	$\dfrac{60}{3+c}$

Table 2. Calculating the time column entries.

To set up an equation, we need to use the fact that the time to travel upstream is twice the time to travel downstream. This leads to the result

$$\frac{60}{3-c} = 2\left(\frac{60}{3+c}\right),$$

or equivalently,

$$\frac{60}{3-c} = \frac{120}{3+c}.$$

Multiply both sides by the common denominator, in this case, $(3-c)(3+c)$.

$$\begin{aligned}(3-c)(3+c)\left[\frac{60}{3-c}\right] &= \left[\frac{120}{3+c}\right](3-c)(3+c)\\ 60(3+c) &= 120(3-c)\end{aligned}$$

Expand each side of this equation.

$$180 + 60c = 360 - 120c$$

This equation is *linear* (no power of c other than 1). Hence, we want to isolate all terms containing c on one side of the equation. We add $120c$ to both sides of the equation, then subtract 180 from both sides of the equation.

$$60c + 120c = 360 - 180$$

From here, it is simple to solve for c.

$$\begin{aligned}180c &= 180\\ c &= 1.\end{aligned}$$

Hence, the speed of the current is 1 mile per hour.

It is important to check that the solution satisfies the constraints of the problem statement.

- If the speed of the boat in still water is 3 miles per hour and the speed of the current is 1 mile per hour, then the speed of the boat upstream (against the current) will be 2 miles per hour. It will take 30 hours to travel 60 miles at this rate.
- The speed of the boat as it goes downstream (with the current) will be 4 miles per hour. It will take 15 hours to travel 60 miles at this rate.

Note that the time to travel upstream (30 hours) is twice the time to travel downstream (15 hours), so our solution is correct.

Let's look at another example.

I **Example 6.** *A speedboat can travel 32 miles per hour in still water. It travels 150 miles upstream against the current then returns to the starting location. The total time of the trip is 10 hours. What is the speed of the current?*

Let c represent the speed of the current. Going upstream, the boat struggles against the current, so its net speed is $32-c$ miles per hour. On the return trip, the boat benefits from the current, so its net speed on the return trip is $32+c$ miles per hour. The trip each way is 150 miles. We've entered this data in **Table 3**.

	d (mi)	v (mi/h)	t (h)
Upstream	150	$32 - c$	?
Downstream	150	$32 + c$	?

Table 3. Entering the given data in a distance, speed, and time table.

Solving $d = vt$ for the time t,

$$t = \frac{d}{v}.$$

In the first row of **Table 3**, we have $d = 150$ miles and $v = 32 - c$ miles per hour. Hence, the time it takes the boat to go upstream is given by

$$t = \frac{d}{v} = \frac{150}{32 - c}.$$

Similarly, upon examining the data in the second row of **Table 3**, the time it takes the boat to return downstream to its starting location is

$$t = \frac{d}{v} = \frac{150}{32 + c}.$$

These results are entered in **Table 4**.

	d (mi)	v (mi/h)	t (h)
Upstream	150	$32 - c$	$150/(32 - c)$
Downstream	150	$32 + c$	$150/(32 + c)$

Table 4. Calculating the time to go upstream and return.

Because the total time to go upstream and return is 10 hours, we can write

$$\frac{150}{32 - c} + \frac{150}{32 + c} = 10.$$

Multiply both sides by the common denominator $(32 - c)(32 + c)$.

$$(32 - c)(32 + c)\left(\frac{150}{32 - c} + \frac{150}{32 + c}\right) = 10(32 - c)(32 + c)$$
$$150(32 + c) + 150(32 - c) = 10(1024 - c^2)$$

We can make the numbers a bit smaller by noting that both sides of the last equation are divisible by 10.

$$15(32 + c) + 15(32 - c) = 1024 - c^2$$

Expand, simplify, make one side zero, then factor.

$$\begin{aligned} 480 + 15c + 480 - 15c &= 1024 - c^2 \\ 960 &= 1024 - c^2 \\ 0 &= 64 - c^2 \\ 0 &= (8+c)(8-c) \end{aligned}$$

Using the zero product property, either

$$8 + c = 0 \qquad \text{or} \qquad 8 - c = 0,$$

providing two solutions for the current,

$$c = -8 \qquad \text{or} \qquad c = 8.$$

Discarding the negative answer (speed is a positive quantity in this case), the speed of the current is 8 miles per hour.

Does our answer make sense?

- Because the speed of the current is 8 miles per hour, the boat travels 150 miles upstream at a net speed of 24 miles per hour. This will take 150/24 or 6.25 hours.
- The boat travels downstream 150 miles at a net speed of 40 miles per hour. This will take 150/40 or 3.75 hours.

Note that the total time to go upstream and return is $6.25 + 3.75$, or 10 hours.

Let's look at another class of problems.

Work Problems

A nice application of rational functions involves the amount of work a person (or team of persons) can do in a certain amount of time. We can handle these applications involving work in a manner similar to the method we used to solve distance, speed, and time problems. Here is the guiding principle.

> **Work, Rate, and Time.** The amount of work done is equal to the product of the rate at which work is being done and the amount of time required to do the work. That is,
>
> $$\text{Work} = \text{Rate} \times \text{Time}.$$

For example, suppose that Emilia can mow lawns at a rate of 3 lawns per hour. After 6 hours,

$$\text{Work} = 3\,\frac{\text{lawns}}{\text{hr}} \times 6\,\text{hr} = 18\,\text{lawns}.$$

A second important concept is the fact that rates add. For example, if Emilia can mow lawns at a rate of 3 lawns per hour and Michele can mow the same lawns at a rate of 2 lawns per hour, then together they can mow the lawns at a combined rate of 5 lawns per hour.

Let's look at an example.

I **Example 7.** *Bill can finish a report in 2 hours. Maria can finish the same report in 4 hours. How long will it take them to finish the report if they work together?*

A common misconception is that the times add in this case. That is, it takes Bill 2 hours to complete the report and it takes Maria 4 hours to complete the same report, so if Bill and Maria work together it will take 6 hours to complete the report. A little thought reveals that this result is nonsense. Clearly, if they work together, it will take them less time than it takes Bill to complete the report alone; that is, the combined time will surely be less than 2 hours.

However, as we saw above, the rates at which they are working will add. To take advantage of this fact, we set up what we know in a Work, Rate, and Time table (see **Table 5**).

- It takes Bill 2 hours to complete 1 report. This is reflected in the entries in the first row of **Table 5**.
- It takes Maria 4 hours to complete 1 report. This is reflected in the entries in the second row of **Table 5**.
- Let t represent the time it takes them to complete 1 report if they work together. This is reflected in the entries in the last row of **Table 5**.

	w (reports)	r (reports/h)	t (h)
Bill	1	?	2
Maria	1	?	4
Together	1	?	t

Table 5. A work, rate, and time table.

We have advice similar to that given for distance, speed, and time tables.

Work, Rate, and Time Tables. Because work, rate, and time are related by the equation

$$\text{Work} = \text{Rate} \times \text{Time},$$

whenever you have two boxes in a row completed, the third box in that row can be calculated by means of the relation Work = Rate × Time.

In the case of **Table 5**, we can calculate the rate at which Bill is working by solving the equation Work = Rate × Time for the Rate, then substitute Bill's data from row one of **Table 5**.

$$\text{Rate} = \frac{\text{Work}}{\text{Time}} = \frac{1\,\text{report}}{2\,\text{h}}.$$

Thus, Bill is working at a rate of 1/2 report per hour. Note how we've entered this result in the first row of **Table 6**. Similarly, Maria is working at a rate of 1/4 report per hour, which we've also entered in **Table 6**.

We've let t represent the time it takes them to write 1 report if they are working together (see **Table 5**), so the following calculation gives us the combined rate.

$$\text{Rate} = \frac{\text{Work}}{\text{Time}} = \frac{1\,\text{report}}{t\,\text{h}}.$$

That is, together they work at a rate of $1/t$ reports per hour. This result is also recorded in **Table 6**.

	w (reports)	r (reports/h)	t (h)
Bill	1	1/2	2
Maria	1	1/4	4
Together	1	$1/t$	t

Table 6. Calculating the Rate entries.

In our discussion above, we pointed out the fact that rates add. Thus, the equation we seek lies in the Rate column of **Table 6**. Bill is working at a rate of 1/2 report per hour and Maria is working at a rate of 1/4 report per hour. Therefore, their combined rate is $1/2+1/4$ reports per hour. However, the last row of **Table 6** indicates that the combined rate is also $1/t$ reports per hour. Thus,

$$\frac{1}{2} + \frac{1}{4} = \frac{1}{t}.$$

Multiply both sides of this equation by the common denominator $4t$.

$$\begin{aligned} (4t)\left[\frac{1}{2} + \frac{1}{4}\right] &= \left[\frac{1}{t}\right](4t) \\ 2t + t &= 4, \end{aligned}$$

This equation is linear (no power of t other than 1) and is easily solved.

$$\begin{aligned} 3t &= 4 \\ t &= 4/3 \end{aligned}$$

Thus, it will take 4/3 of an hour to complete 1 report if Bill and Maria work together.

Again, it is very important that we check this result.

- We know that Bill does 1/2 reports per hour. In 4/3 of an hour, Bill will complete

  $$\text{Work} = \frac{1}{2}\,\frac{\text{reports}}{\text{h}} \times \frac{4}{3}\,\text{h} = \frac{2}{3}\,\text{reports}.$$

  That is, Bill will complete 2/3 of a report.
- We know that Maria does 1/4 reports per hour. In 4/3 of an hour, Maria will complete

$$\text{Work} = \frac{1}{4}\,\frac{\text{reports}}{\text{h}} \times \frac{4}{3}\,\text{h} = \frac{1}{3}\,\text{reports}.$$

That is, Maria will complete $1/3$ of a report.

Clearly, working together, Bill and Maria will complete $2/3 + 1/3$ reports, that is, one full report.

Let's look at another example.

I **Example 8.** *It takes Liya 7 more hours to paint a kitchen than it takes Hank to complete the same job. Together, they can complete the same job in 12 hours. How long does it take Hank to complete the job if he works alone?*

Let H represent the time it take Hank to complete the job of painting the kitchen when he works alone. Because it takes Liya 7 more hours than it takes Hank, let $H + 7$ represent the time it takes Liya to paint the kitchen when she works alone. This leads to the entries in **Table 7**.

	w (kitchens)	r (kitchens/h)	t (h)
Hank	1	?	H
Liya	1	?	$H + 7$
Together	1	?	12

Table 7. Entering the given data for Hank and Liya.

We can calculate the rate at which Hank is working alone by solving the equation Work = Rate × Time for the Rate, then substituting Hank's data from row one of **Table 7**.

$$\text{Rate} = \frac{\text{Work}}{\text{Time}} = \frac{1\,\text{kitchen}}{H\,\text{hour}}$$

Thus, Hank is working at a rate of $1/H$ kitchens per hour. Similarly, Liya is working at a rate of $1/(H + 7)$ kitchens per hour. Because it takes them 12 hours to complete the task when working together, their combined rate is $1/12$ kitchens per hour. Each of these rates is entered in **Table 8**.

	w (kitchens)	r (kitchens/h)	t (h)
Hank	1	$1/H$	H
Liya	1	$1/(H + 7)$	$H + 7$
Together	1	$1/12$	12

Table 8. Calculating the rates.

Because the rates add, we can write

$$\frac{1}{H} + \frac{1}{H+7} = \frac{1}{12}.$$

Multiply both sides of this equation by the common denominator $12H(H+7)$.

$$12H(H+7)\left(\frac{1}{H}+\frac{1}{H+7}\right)=\left(\frac{1}{12}\right)12H(H+7)$$
$$12(H+7)+12H=H(H+7)$$

Expand and simplify.

$$\begin{aligned}12H+84+12H&=H^2+7H\\24H+84&=H^2+7H\end{aligned}$$

This last equation is nonlinear, so make one side zero by subtracting $24H$ and 84 from both sides of the equation.

$$\begin{aligned}0&=H^2+7H-24H-84\\0&=H^2-17H-84\end{aligned}$$

Note that $ac=(1)(-84)=-84$. The integer pair $\{4,-21\}$ has product -84 and sums to -17. Hence,

$$0=(H+4)(H-21).$$

Using the zero product property, either

$$H+4=0 \qquad \text{or} \qquad H-21=0,$$

leading to the solutions

$$H=-4 \qquad \text{or} \qquad H=21.$$

We eliminate the solution $H=-4$ from consideration (it doesn't take Hank negative time to paint the kitchen), so we conclude that it takes Hank 21 hours to paint the kitchen.

Does our solution make sense?

- It takes Hank 21 hours to complete the kitchen, so he is finishing $1/21$ of the kitchen per hour.
- It takes Liya 7 hours longer than Hank to complete the kitchen, namely 28 hours, so she is finishing $1/28$ of the kitchen per hour.

Together, they are working at a combined rate of

$$\frac{1}{21}+\frac{1}{28}=\frac{4}{84}+\frac{3}{84}=\frac{7}{84}=\frac{1}{12},$$

or $1/12$ of a kitchen per hour. This agrees with the combined rate in **Table 8**.

3.7 *Exercises*

1. The sum of the reciprocals of two consecutive odd integers is $-\frac{16}{63}$. Find the two numbers.

2. The sum of the reciprocals of two consecutive odd integers is $\frac{28}{195}$. Find the two numbers.

3. The sum of the reciprocals of two consecutive integers is $-\frac{19}{90}$. Find the two numbers.

4. The sum of a number and its reciprocal is $\frac{41}{20}$. Find the number(s).

5. The sum of the reciprocals of two consecutive even integers is $\frac{5}{12}$. Find the two numbers.

6. The sum of the reciprocals of two consecutive integers is $\frac{19}{90}$. Find the two numbers.

7. The sum of a number and twice its reciprocal is $\frac{9}{2}$. Find the number(s).

8. The sum of a number and its reciprocal is $\frac{5}{2}$. Find the number(s).

9. The sum of the reciprocals of two consecutive even integers is $\frac{11}{60}$. Find the two numbers.

10. The sum of a number and twice its reciprocal is $\frac{17}{6}$. Find the number(s).

11. The sum of the reciprocals of two numbers is 15/8, and the second number is 2 larger than the first. Find the two numbers.

12. The sum of the reciprocals of two numbers is 16/15, and the second number is 1 larger than the first. Find the two numbers.

13. Moira can paddle her kayak at a speed of 2 mph in still water. She paddles 3 miles upstream against the current and then returns to the starting location. The total time of the trip is 9 hours. What is the speed (in mph) of the current? Round your answer to the nearest hundredth.

14. Boris is kayaking in a river with a 6 mph current. Suppose that he can kayak 4 miles upstream in the same amount of time as it takes him to kayak 9 miles downstream. Find the speed (mph) of Boris's kayak in still water.

15. Jacob can paddle his kayak at a speed of 6 mph in still water. He paddles 5 miles upstream against the current and then returns to the starting location. The total time of the trip is 5 hours. What is the speed (in mph) of the current? Round your answer to the nearest hundredth.

16. Boris can paddle his kayak at a speed of 6 mph in still water. If he can paddle 5 miles upstream in the same amount of time as it takes him to paddle 9 miles downstream, what is the speed of the current?

17. Jacob is canoeing in a river with a 5 mph current. Suppose that he can canoe 4 miles upstream in the same amount of time as it takes him to canoe 8 miles downstream. Find the speed (mph) of Jacob's canoe in still water.

18. The speed of a freight train is 16 mph slower than the speed of a passenger train. The passenger train travels 518 miles in the same time that the freight train travels 406 miles. Find the speed of the freight train.

19. The speed of a freight train is 20 mph slower than the speed of a passenger train. The passenger train travels 440 miles in the same time that the freight train travels 280 miles. Find the speed of the freight train.

20. Emily can paddle her canoe at a speed of 2 mph in still water. She paddles 5 miles upstream against the current and then returns to the starting location. The total time of the trip is 6 hours. What is the speed (in mph) of the current? Round your answer to the nearest hundredth.

21. Jacob is canoeing in a river with a 2 mph current. Suppose that he can canoe 2 miles upstream in the same amount of time as it takes him to canoe 5 miles downstream. Find the speed (mph) of Jacob's canoe in still water.

22. Moira can paddle her kayak at a speed of 2 mph in still water. If she can paddle 4 miles upstream in the same amount of time as it takes her to paddle 8 miles downstream, what is the speed of the current?

23. Boris can paddle his kayak at a speed of 6 mph in still water. If he can paddle 5 miles upstream in the same amount of time as it takes him to paddle 10 miles downstream, what is the speed of the current?

24. The speed of a freight train is 19 mph slower than the speed of a passenger train. The passenger train travels 544 miles in the same time that the freight train travels 392 miles. Find the speed of the freight train.

25. It takes Jean 15 hours longer to complete an inventory report than it takes Sanjay. If they work together, it takes them 10 hours. How many hours would it take Sanjay if he worked alone?

26. Jean can paint a room in 5 hours. It takes Amelie 10 hours to paint the same room. How many hours will it take if they work together?

27. It takes Amelie 18 hours longer to complete an inventory report than it takes Jean. If they work together, it takes them 12 hours. How many hours would it take Jean if she worked alone?

28. Sanjay can paint a room in 5 hours. It takes Amelie 9 hours to paint the same room. How many hours will it take if they work together?

29. It takes Ricardo 12 hours longer to complete an inventory report than it takes Sanjay. If they work together, it takes them 8 hours. How many hours would it take Sanjay if he worked alone?

30. It takes Ricardo 8 hours longer to complete an inventory report than it takes Amelie. If they work together, it takes them 3 hours. How many hours would it take Amelie if she worked alone?

31. Jean can paint a room in 4 hours. It takes Sanjay 7 hours to paint the same room. How many hours will it take if they work together?

32. Amelie can paint a room in 5 hours. It takes Sanjay 9 hours to paint the same room. How many hours will it take if they work together?

3.7 *Answers*

1. $-9, -7$

3. $-10, -9$

5. $4, 6$

7. $\frac{1}{2}, 4$

9. $10, 12$

11. $\{2/3, 8/3\}$ and $\{-8/5, 2/5\}$

13. 1.63 mph

15. 4.90 mph

17. 15 mph

19. 35 mph

21. $\frac{14}{3}$ mph

23. 2 mph

25. 15 hours

27. 18 hours

29. 12 hours

31. $\frac{28}{11}$ hours

3.8 Direct and Inverse Variation

We start with the definition of the phrase “is proportional to.”

Proportional. We say that y is *proportional* to x if and only if

$$y = kx,$$

where k is a constant called the *constant of proportionality*. The phrase “y *varies directly* as x” is an equivalent way of saying “y is proportional to x.”

Here are a few examples that translate the phrase “is proportional to.”

- Given that d is proportional to t, we write $d = kt$, where k is a constant.
- Given that y is proportional to the cube of x, we write $y = kx^3$, where k is a constant.
- Given that s is proportional to the square of t, we write $s = kt^2$, where k is a constant.

We are not restricted to always using the letter k for our constant of proportionality.

You Try It!

Given that y is proportional to x and that $y = 21$ when $x = 9$, determine the value of y when $x = 27$.

EXAMPLE 1. Given that y is proportional to x and the fact that $y = 12$ when $x = 5$, determine the constant of proportionality, then determine the value of y when $x = 10$.

Solution: Given the fact the y is proportional to x, we know immediately that

$$y = kx,$$

where k is the proportionality constant. Because we are given that $y = 12$ when $x = 5$, we can substitute 12 for y and 5 for x to determine k.

$$\begin{aligned} y &= kx && \text{y is proportional to x.} \\ 12 &= k(5) && \text{Substitute 12 for y, 5 for x.} \\ \frac{12}{5} &= k && \text{Divide both sides by 5.} \end{aligned}$$

Next, substitute the constant of proportionality 12/5 for k in $y = kx$, then substitute 10 for x to determine y when $x = 10$.

$$\begin{aligned} y &= \frac{12}{5}x && \text{Substitute 12/5 for k.} \\ y &= \frac{12}{5}(10) && \text{Substitute 10 for x.} \\ y &= 24 && \text{Cancel and simplify.} \end{aligned}$$

Answer: 63

□

You Try It!

A ball is dropped from the edge of a cliff on a certain planet. The distance s the ball falls is proportional to the square of the time t that has passed since the ball's release. If the ball falls 50 feet during the first 5 seconds, how far does the ball fall in 8 seconds?

EXAMPLE 2. A ball is dropped from a balloon floating above the surface of the earth. The distance s the ball falls is proportional to the square of the time t that has passed since the ball's release. If the ball falls 144 feet during the first 3 seconds, how far does the ball fall in 9 seconds?

Solution: Given the fact the s is proportional to the square of t, we know immediately that

$$s = kt^2,$$

where k is the proportionality constant. Because we are given that the ball falls 144 feet during the first 3 seconds, we can substitute 144 for s and 3 for t to determine the constant of proportionality.

$s = kt^2$	s is proportional to the square of t.
$144 = k(3)^2$	Substitute 144 for s, 3 for t.
$144 = 9k$	Simplify: $3^2 = 9$.
$16 = k$	Divide both sides by 9.

Next, substitute the constant of proportionality 16 for k in $s = kt^2$, and then substitute 9 for t to determine the distance fallen when $t = 9$ seconds.

$s = 16t^2$	Substitute 16 for k.
$s = 16(9)^2$	Substitute 9 for t.
$s = 1296$	Simplify.

Thus, the ball falls 1,296 feet during the first 9 seconds.

Answer: 128 feet

□

You Try It!

If a 0.75 pound weight stretches a spring 5 inches, how far will a 1.2 pound weight stretch the spring?

EXAMPLE 3. Tony and Paul are hanging weights on a spring in the physics lab. Each time a weight is hung, they measure the distance the spring stretches. They discover that the distance y that the spring stretches is proportional to the weight hung on the spring (Hooke's Law). If a 0.5 pound weight stretches the spring 3 inches, how far will a 0.75 pound weight stretch the spring?

Solution: Let W represent the weight hung on the spring. Let y represent the distance the spring stretches. We're told that the distance y the spring stretches is proportional to the amount of weight W hung on the spring. Hence, we can write:

$y = kW$	y is proportional to W.

Substitute 3 for y, 0.5 for W, then solve for k.

$$\begin{aligned} 3 &= k(0.5) && \text{Substitute 3 for } y, \text{ 0.5 for } W. \\ \frac{3}{0.5} &= k && \text{Divide both sides by 0.5.} \\ k &= 6 && \text{Simplify.} \end{aligned}$$

Substitute 6 for k in $y = kW$ to produce:

$$y = 6W \qquad \text{Substitute 6 for } k \text{ in } y = kW.$$

To determine the distance the spring will stretch when 0.75 pounds are hung on the spring, substitute 0.75 for W.

$$\begin{aligned} y &= 6(0.75) && \text{Substitute 0.75 for } W. \\ y &= 4.5 && \text{Simplify.} \end{aligned}$$

Thus, the spring will stretch 4.5 inches.

Answer: 8 inches

□

Inversely Proportional

In Examples 1, 2, and 3, where one quantity was proportional to a second quantity, you may have noticed that when one quantity increased, the second quantity also increased. Vice-versa, when one quantity decreased, the second quantity also decreased.

However, not all real-world situations follow this pattern. There are times when as one quantity increases, the related quantity decreases. For example, consider the situation where you increase the number of workers on a job and note that the time to finish the job decreases. This is an example of a quantity being *inversely proportional* to a second quantity.

Inversely proportional. We say the y is *inversely proportional* to x if and only if

$$y = \frac{k}{x},$$

where k is a constant called the *constant of proportionality*. The phrase "y *varies inversely* as x" is an equivalent way of saying "y in inversely proportional to x."

Here are a few examples that translate the phrase "is inversely proportional to."

- Given that d is inversely proportional to t, we write $d = k/t$, where k is a constant.

- Given that y is inversely proportional to the cube of x, we write $y = k/x^3$, where k is a constant.
- Given that s is inversely proportional to the square of t, we write $s = k/t^2$, where k is a constant.

We are not restricted to always using the letter k for our constant of proportionality.

You Try It!

Given that y is inversely proportional to x and that $y = 5$ when $x = 8$, determine the value of y when $x = 10$.

EXAMPLE 4. Given that y is inversely proportional to x and the fact that $y = 4$ when $x = 2$, determine the constant of proportionality, then determine the value of y when $x = 4$.

Solution: Given the fact the y is inversely proportional to x, we know immediately that

$$y = \frac{k}{x},$$

where k is the proportionality constant. Because we are given that $y = 4$ when $x = 2$, we can substitute 4 for y and 2 for x to determine k.

$$\begin{aligned} y &= \frac{k}{x} && \text{y is inversely proportional to x.} \\ 4 &= \frac{k}{2} && \text{Substitute 4 for y, 2 for x.} \\ 8 &= k && \text{Multiply both sides by 2.} \end{aligned}$$

Substitute 8 for k in $y = k/x$, then substitute 4 for x to determine y when $x = 4$.

$$\begin{aligned} y &= \frac{8}{x} && \text{Substitue 8 for k.} \\ y &= \frac{8}{4} && \text{Substitute 4 for x.} \\ y &= 2 && \text{Reduce.} \end{aligned}$$

Note that as x increased from 2 to 4, y decreased from 4 to 2. □

Answer: 4

You Try It!

If the light intensity 4 feet from a light source is 2 foot-candles, what is the intensity of the light 8 feet from the light source?

EXAMPLE 5. The intensity I of light is inversely proportional to the square of the distance d from the light source. If the light intensity 5 feet from the light source is 3 foot-candles, what is the intensity of the light 15 feet from the light source?

Solution: Given the fact that the intensity I of the light is inversely proportional to the square of the distance d from the light source, we know immediately that

$$I = \frac{k}{d^2},$$

where k is the proportionality constant. Because we are given that the intensity is $I = 3$ foot-candles at $d = 5$ feet from the light source, we can substitute 3 for I and 5 for d to determine k.

$$\begin{aligned} I &= \frac{k}{d^2} && \text{I is inversely proportional to d^2.} \\ 3 &= \frac{k}{5^2} && \text{Substitute 3 for I, 5 for d.} \\ 3 &= \frac{k}{25} && \text{Simplify.} \\ 75 &= k && \text{Multiply both sides by 25.} \end{aligned}$$

Substitute 75 for k in $I = k/d^2$, then substitute 15 for d to determine I when $d = 15$.

$$\begin{aligned} I &= \frac{75}{d^2} && \text{Substitute 75 for k.} \\ I &= \frac{75}{15^2} && \text{Substitute 15 for d.} \\ I &= \frac{75}{225} && \text{Simplify.} \\ I &= \frac{1}{3} && \text{Reduce.} \end{aligned}$$

Thus, the intensity of the light 15 feet from the light source is 1/3 foot-candle.

Answer: 1/2 foot-candle

□

You Try It!

Suppose that the price per person for a tour is inversely proportional to the number of people who sign up for the tour. If 8 people sign up, the price per person is \$70. What will be the price per person if 20 people sign up?

EXAMPLE 6. Suppose that the price per person for a camping experience is inversely proportional to the number of people who sign up for the experience. If 10 people sign up, the price per person is \$350. What will be the price per person if 50 people sign up?

Solution: Let p represent the price per person and let N be the number of people who sign up for the camping experience. Because we are told that the price per person is inversely proportional to the number of people who sign up for the camping experience, we can write:

$$p = \frac{k}{N},$$

where k is the proportionality constant. Because we are given that the price per person is \$350 when 10 people sign up, we can substitute 350 for p and 10 for N to determine k.

$$\begin{aligned} p &= \frac{k}{N} && \text{p is inversely proportional to N.} \\ 350 &= \frac{k}{10} && \text{Substitute 350 for p, 10 for N.} \\ 3500 &= k && \text{Multiply both sides by 10.} \end{aligned}$$

Substitute 3500 for k in $p = k/N$, then substitute 50 for N to determine p when $N = 50$.

$$\begin{aligned} p &= \frac{3500}{N} && \text{Substitute 3500 for k.} \\ p &= \frac{3500}{50} && \text{Substitute 50 for N.} \\ p &= 70 && \text{Simplify.} \end{aligned}$$

Answer: \$28

Thus, the price per person is \$70 if 50 people sign up for the camping experience.

□

3.8 *Exercises*

1. Given that s is proportional to t and the fact that $s = 632$ when $t = 79$, determine the value of s when $t = 50$.

2. Given that s is proportional to t and the fact that $s = 264$ when $t = 66$, determine the value of s when $t = 60$.

3. Given that s is proportional to the cube of t and the fact that $s = 1588867$ when $t = 61$, determine the value of s when $t = 63$.

4. Given that d is proportional to the cube of t and the fact that $d = 318028$ when $t = 43$, determine the value of d when $t = 76$.

5. Given that q is proportional to the square of c and the fact that $q = 13448$ when $c = 82$, determine the value of q when $c = 29$.

6. Given that q is proportional to the square of c and the fact that $q = 3125$ when $c = 25$, determine the value of q when $c = 87$.

7. Given that y is proportional to the square of x and the fact that $y = 14700$ when $x = 70$, determine the value of y when $x = 45$.

8. Given that y is proportional to the square of x and the fact that $y = 2028$ when $x = 26$, determine the value of y when $x = 79$.

9. Given that F is proportional to the cube of x and the fact that $F = 214375$ when $x = 35$, determine the value of F when $x = 36$.

10. Given that d is proportional to the cube of t and the fact that $d = 2465195$ when $t = 79$, determine the value of d when $t = 45$.

11. Given that d is proportional to t and the fact that $d = 496$ when $t = 62$, determine the value of d when $t = 60$.

12. Given that d is proportional to t and the fact that $d = 405$ when $t = 45$, determine the value of d when $t = 65$.

13. Given that h is inversely proportional to x and the fact that $h = 16$ when $x = 29$, determine the value of h when $x = 20$.

14. Given that y is inversely proportional to x and the fact that $y = 23$ when $x = 15$, determine the value of y when $x = 10$.

15. Given that q is inversely proportional to the square of c and the fact that $q = 11$ when $c = 9$, determine the value of q when $c = 3$.

16. Given that s is inversely proportional to the square of t and the fact that $s = 11$ when $t = 8$, determine the value of s when $t = 10$.

17. Given that F is inversely proportional to x and the fact that $F = 19$ when $x = 22$, determine the value of F when $x = 16$.

18. Given that d is inversely proportional to t and the fact that $d = 21$ when $t = 16$, determine the value of d when $t = 24$.

19. Given that y is inversely proportional to the square of x and the fact that $y = 14$ when $x = 4$, determine the value of y when $x = 10$.

20. Given that d is inversely proportional to the square of t and the fact that $d = 21$ when $t = 8$, determine the value of d when $t = 12$.

21. Given that d is inversely proportional to the cube of t and the fact that $d = 18$ when $t = 2$, determine the value of d when $t = 3$.

22. Given that q is inversely proportional to the cube of c and the fact that $q = 10$ when $c = 5$, determine the value of q when $c = 6$.

23. Given that q is inversely proportional to the cube of c and the fact that $q = 16$ when $c = 5$, determine the value of q when $c = 6$.

24. Given that q is inversely proportional to the cube of c and the fact that $q = 15$ when $c = 6$, determine the value of q when $c = 2$.

---

25. Joe and Mary are hanging weights on a spring in the physics lab. Each time a weight is hung, they measure the distance the spring stretches. They discover that the distance that the spring stretches is proportional to the weight hung on the spring. If a 2 pound weight stretches the spring 16 inches, how far will a 5 pound weight stretch the spring?

26. Liz and Denzel are hanging weights on a spring in the physics lab. Each time a weight is hung, they measure the distance the spring stretches. They discover that the distance that the spring stretches is proportional to the weight hung on the spring. If a 5 pound weight stretches the spring 12.5 inches, how far will a 12 pound weight stretch the spring?

27. The intensity I of light is inversely proportional to the square of the distance d from the light source. If the light intensity 4 feet from the light source is 20 foot-candles, what is the intensity of the light 18 feet from the light source?

28. The intensity I of light is inversely proportional to the square of the distance d from the light source. If the light intensity 5 feet from the light source is 10 foot-candles, what is the intensity of the light 10 feet from the light source?

29. Suppose that the price per person for a camping experience is inversely proportional to the number of people who sign up for the experience. If 18 people sign up, the price per person is \$204. What will be the price per person if 35 people sign up? Round your answer to the nearest dollar.

30. Suppose that the price per person for a camping experience is inversely proportional to the number of people who sign up for the experience. If 17 people sign up, the price per person is \$213. What will be the price per person if 27 people sign up? Round your answer to the nearest dollar.

3.8 *Answers*

1. 400

3. 1750329

5. 1682

7. 6075

9. 233280

11. 480

13. 116/5

15. 99

17. 209/8

19. 56/25

21. 16/3

23. 250/27

25. 40 inches

27. 1.0 foot-candles

29. $105

Chapter 4 Exponential and Logarithmic Functions

In this chapter, we will investigate two more families of functions: exponential functions and logarithmic functions. These are two of the most important functions in mathematics, and both types of functions are used extensively in the study of real-world phenomena. In particular, a good understanding of the concepts of exponential growth and decay is necessary for students in both the natural and social sciences.

Our main focus will be on the nature of exponential functions, and their use in describing and solving problems involving compound interest, population growth, and radioactive decay. Our work with logarithmic functions will be a more limited introduction, mostly concentrating on their relationship with exponential functions and their use in solving exponential equations.

4.1 Introduction to Exponential Functions

Objectives

- To describe an exponential function in terms of an initial value and a multiplier applied over an interval.
- To use the description to write a sentence describing the function.
- To use the description to write a formula for the function of the form $f(x) = a \cdot M^{x/k}$.

Background: Recall that describing a linear function required the following three values: (1) the y-coordinate of the y-intercept to indicate a "starting point" for the function, (2) the rise to indicate how much the function goes up or down, and (3) the run to indicate how often the rise takes effect. Also recall that we combine the rise and the run into one number called the slope. Thus, we have a value to indicate **where the function starts**, and another two values to indicate **how the function changes**.

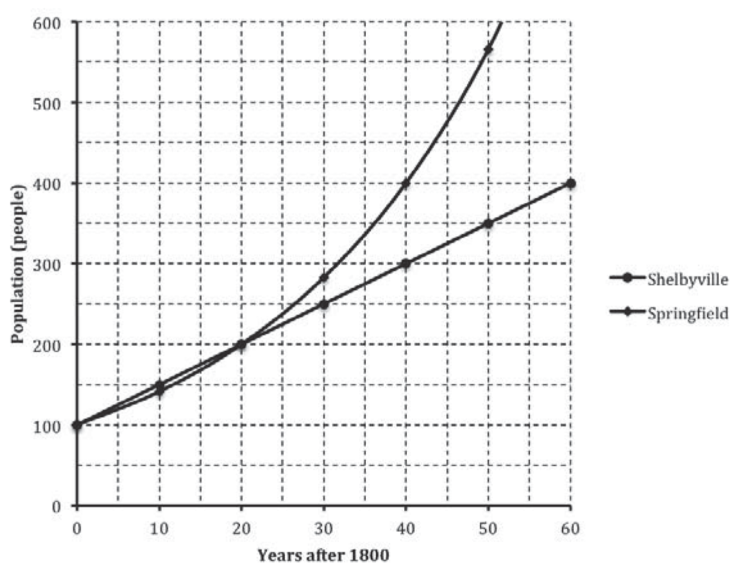
Overview: Similarly, to describe an exponential function, we need to find the following three values: (1) the y-coordinate of the y-intercept to indicate a "starting point" for the function, (2) a multiplier to indicate how the values of the function are changing, and (3) an interval to indicate how often the multiplier is applied to the values of the function. Thus, we have a value to indicate **where the function starts** and another two values to indicate **how the function changes**.

Recall that exponents are used to denote repeated multiplication. This is why functions following a pattern of repeated multiplication are exponential functions.

► **Example 1.** *Consider the towns of Shelbyville and Springfield, both of which were founded in* 1800 *with populations of* 100 *people. The populations of Shelbyville and Springfield for various years are given in the table.*

Year	**1800**	1810	**1820**	1830	**1840**	1850
Shelbyville	**100**	150	**200**	250	**300**	350
Springfield	**100**	141	**200**	283	**400**	566

Let $P(x)$ be the population of Shelbyville at x years after 1800 and let $Q(x)$ be the population of Springfield at x years after 1800. Note that $P(0) = 100$ and $Q(0) = 100$ and that $P(20) = 200$ and $Q(20) = 200$. That is, the populations of Shelbyville and Springfield are the same in 1800 and 1820. After another 20 years however, we see that the population of Shelbyville has grown to 300 while the population of Springfield has grown to 400. That is, $P(40) = 300$ and $Q(40) = 400$.



Notice that the graph of the population of Shelbyville is a line due to the constant rate of change. The graph of the population of Springfield, however, does not have a constant rate of change and thus is curved upward.

We see a pattern of repeated addition in the population of Shelbyville. So we describe how the population of Shelbyville is changing using the slope, or the rise divided by the run.

- The population of Shelbyville was 100 in 1800 and increased by 100 people every 20 years.

Recall that we usually simplify the slope to lowest terms, which gives us the following equivalent statement:

- The population of Shelbyville was 100 in 1800 and increased by 5 people every year.

A linear model for the population of Shelbyville is given by $P(x) = 5x + 100$.

The population of Springfield is changing with a pattern of repeated multiplication. Specifically, we observe the following:

- The population of Springfield is multiplied by 2 every 20 years.

Now we need to write a formula for the population of Springfield. Recall that exponents denote repeated multiplication. This is why functions following a pattern of repeated multiplication are exponential functions.

Since the population of Springfield is multiplied by 2 every 20 years, we need a formula that multiplies by two once after 20 years, twice after 40 years (i.e. two twenties), three times after 60 years (i.e. three twenties), and so on. For this, we will need the following notation.

Notation: We will use the following notation for the values we use in describing an exponential function:

- The y-intercept of the function will be $(0, a)$,
- the multiplier will be denoted M, and
- the interval will be denoted k.

We will describe the exponential function $f(x)$ using these values by writing the sentence

$$f(0) = a \text{ and } f(x) \text{ is multiplied by } M \text{ every } k \text{ units.}$$

We will describe the exponential function $f(x)$ using these values by writing the formula

$$f(x) = a \cdot M^{x/k}.$$

In the population of Springfield, $a = 100$ since the y-intercept is $(0, 100)$, $M = 2$ and $k = 20$ since the population of Springfield is multiplied by 2 every 20 years. For any value of x, we see that $x/20$ is the number of times $Q(x)$ is multiplied by 2, as shown in the table. Thus a formula for the population of Springfield is given by $Q(x) = 100 \cdot (2)^{x/20}$.

x	0	20	40	60	80	100
x/k	0	1	2	3	4	5
$Q(x)$	100	200	400	800	1600	3200

► **Example 2.** *Consider the function f(x), some values of which are shown in the table below. Notice the pattern being followed in both x and f(x), and fill in the final four boxes in the table.*

x	0	3	6	9				
$f(x)$	10	20	40	80				

Notice that the y-intercept of $f(x)$ is $(0, 10)$, so we will say $a = 10$. Notice that whenever x increases by 3 units, $f(x)$ is multiplied by 2. This is a **pattern of repeated multiplication.**

We can describe $f(x)$ with the following sentence:

$$\boxed{f(0) = 10 \text{ and } f(x) \text{ is multiplied by 2 every 3 units.}}$$

A formula for $f(x)$ is given by $\boxed{f(x) = 10(2)^{x/3}.}$

Recall that exponents denote repeated multiplication. This is why functions following a pattern of repeated multiplication are exponential functions. In the previous example, $f(x)$ is multiplied by 2 every 3 units. Thus, if $x = 3$ (i.e. **one** three), then $f(x)$ has been multiplied by 2 **once**. If $x = 6$ (i.e. **two** threes), then $f(x)$ has been multiplied by 2 **twice**. If $x = 9$ (i.e. **three** threes), then $f(x)$ has been multiplied by 2 **three times**.

► **Example 3.** *Consider the function $g(x)$, some values of which are shown in the table below. Notice the pattern being followed in both x and $g(x)$, and fill in the remaining boxes in the table.*

x	-4	0	4	8		
$g(x)$		80	40	20		

We see that whenever x increases by 4 units, $g(x)$ is multiplied by $1/2$. The y-intercept $(0, 80)$ gives the initial value of 80, the multiplier is $1/2$ (or 0.5), and the interval is 4.
We can describe $g(x)$ with the following sentence:

$$\boxed{g(0) = 80 \text{ and } g(x) \text{ is multiplied by } 0.5 \text{ every } 4 \text{ units.}}$$

A formula for $g(x)$ is given by $\boxed{g(x) = 80(0.5)^{x/4}.}$

You may have described g(x) by writing "g(x) is divided by two every four units". Note that the formula requires the description in terms of a multiplier, not a divisor.

Practice Problems

In Practice Problem 1, consider the exponential function $f(x)$.

A.	Complete the table of values for $f(x)$.
B.	Complete the sentence describing $f(x)$ $f(0) =$ _____ and $f(x)$ is multiplied by _____ every _____ units.
C.	Find a formula for $f(x)$ of the form $f(x) = a \cdot M^{x/k}$.

x		0	6			
$f(x)$		20	60			

In Practice Problem 2, consider the exponential function $g(x)$.

A.	Complete the table of values for $g(x)$.
B.	Complete the sentence describing $g(x)$ $g(0) =$ _____ and $g(x)$ is multiplied by _____ every _____ units.
C.	Find a formula for $g(x)$ of the form $g(x) = a \cdot M^{x/k}$.

x			0	7		
$g(x)$			20	10		

In Practice Problem 3, consider the exponential function $P(x)$.

A. Complete the table of values for $P(x)$.

x		0	4			
$P(x)$		20	30			

B. Write a sentence describing $P(x)$. You know, how we do.

C. Find a formula for $P(x)$ of the form $P(x) = a \cdot M^{x/k}$.

In Practice Problem 4, consider the exponential function $Q(x)$.

A. Complete the table of values for $Q(x)$.

x		0	4			
$Q(x)$		40	30			

B. Write a sentence describing $Q(x)$.

C. Find a formula for $Q(x)$ of the form $Q(x) = a \cdot M^{x/k}$.

4.1 Exercises

In Exercises 1-16, all functions are exponential functions. Round your answers to the nearest hundredth if necessary.

In Exercises 1-4, use the partially completed table of values for the function to (a) complete the table of values for the function, (b) write a sentence describing the function and (c) write a formula for the function of the form $f(x) = a \cdot M^{x/k}$.

1.

x		0	3			
$F(x)$		10	20			

2.

x		0	4			
$G(x)$		40	50			

3.

x		0	5	10		
$p(x)$			50	25		

4.

x		0	6			
$q(x)$		80	60			

In Exercises 5-8, use the sentence describing the function to (a) create a table of values containing at least six ordered pairs for the function and (b) write a formula for the function of the form $f(x) = a \cdot M^{x/k}$.

5. $f(0) = 10$ and $f(x)$ is multiplied by 3 every 4 units.

6. $g(0) = 80$ and $g(x)$ is multiplied by 0.5 every 5 units.

7. $P(0) = 20$ and $P(x)$ is multiplied by 4 every 6 units.

8. $Q(0) = 30$ and $Q(x)$ is multiplied by 0.8 every 7 units.

In Exercises 9-12, use the formula for the function to (a) create a table of values containing at least six ordered pairs for the function and (b) write a sentence describing the function.

9. $F(x) = 10(0.6)^{x/5}$

10. $G(x) = 20(3)^{x/6}$

11. $p(x) = 30(1.5)^{x/7}$

12. $q(x) = 40(0.2)^{x/8}$

In Exercises 13-16, decide if the statement is TRUE or FALSE. Explain your answer.

13. If $f(x)$ is multiplied by 4 every 12 units, then $f(x)$ is multiplied by 2 every 6 units.

14. If $g(x)$ is multiplied by 6 every 12 units, then $g(x)$ is multiplied by 3 every 6 units.

15. If $P(x)$ is multiplied by 3 every 6 units, then $P(x)$ is multiplied by 9 every 12 units.

16. If $Q(x)$ is multiplied by 6 every 6 units, then $Q(x)$ is multiplied by 12 every 12 units.

Solutions to Practice Problems

In Practice Problem 1, consider the exponential function $f(x)$.

A. Complete the table of values for $f(x)$.

x	-6	0	6	12	18	24
$f(x)$	6.67	20	60	180	540	1620

B. Complete the sentence describing $f(x)$

$f(0) =$ 20 and $f(x)$ is multiplied by 3 every 6 units.

C. Find a formula for $f(x)$ of the form $f(x) = a \cdot M^{x/k}$.

$$f(x) = 20(3)^{x/6}$$

In Practice Problem 2, consider the exponential function $g(x)$.

A. Complete the table of values for $g(x)$.

x	-14	-7	0	7	14	21
$g(x)$	80	40	20	10	5	2.5

B. Complete the sentence describing $g(x)$

$g(0) =$ 20 and $g(x)$ is multiplied by 0.5 every 7 units.

C. Find a formula for $g(x)$ of the form $g(x) = a \cdot M^{x/k}$.

$$g(x) = 20(0.5)^{x/7}$$

In Practice Problem 3, consider the exponential function $P(x)$.

A. Complete the table of values for $P(x)$.

x	-4	0	4	8	12	16
$P(x)$	13.33	20	30	45	67.5	101.25

B. Write a sentence describing $P(x)$. You know, how we do.

$P(0) =$ _20_ and $P(x)$ is multiplied by _1.5_ every _4_ units.

C. Find a formula for $P(x)$ of the form $P(x) = a \cdot M^{x/k}$.

$$P(x) = 20(1.5)^{x/4}$$

In Practice Problem 4, consider the exponential function $Q(x)$.

A. Complete the table of values for $Q(x)$.

x	-4	0	4	8	12	16
$Q(x)$	53.33	40	30	22.5	16.88	12.66

B. Write a sentence describing $Q(x)$.

$Q(0) =$ _40_ and $Q(x)$ is multiplied by _0.75_ every _4_ units.

C. Find a formula for $Q(x)$ of the form $Q(x) = a \cdot M^{x/k}$.

$$Q(x) = 40(0.75)^{x/4}$$

4.1 Answers

1.

x	-3	0	3	6	9	12
$F(x)$	5	10	20	40	80	160

$F(0) = 10$ and $F(x)$ is multiplied by 2 every 3 units.
$F(x) = 10(2)^{x/3}$.

3.

x	-5	0	5	10	20	30
$p(x)$	200	100	50	25	12.5	6.25

$p(0) = 100$ and $p(x)$ is multiplied by 0.5 every 5 units.
$p(x) = 100(0.5)^{x/5}$.

5.

x	-4	0	4	8	12	16
$f(x)$	3.33	10	30	90	270	810

$f(x) = 10(3)^{x/4}$.

7.

x	-6	0	6	12	18	24
$P(x)$	5	20	80	320	1280	5120

$P(x) = 20(4)^{x/6}$.

9.

x	-5	0	5	10	15	20
$F(x)$	16.67	10	6	3.6	2.16	1.296

$F(0) = 10$ and $F(x)$ is multiplied by 0.6 every 5 units.

11.

x	-7	0	7	14	21	28
$p(x)$	20	30	45	67.5	101.25	151.88

$p(0) = 30$ and $p(x)$ is multiplied by 1.5 every 7 units.

13. TRUE

15. TRUE

Hint for Exercises 13-16: Create a table of values for each description and compare the results for your explanation.

4.2 Exponents and Roots

Before defining the next family of functions, the *exponential functions*, we will need to discuss exponent notation in detail. As we shall see, exponents can be used to describe not only powers (such as 5^2 and 2^3), but also roots (such as square roots and cube roots). Along the way, we'll define higher roots and develop a few of their properties. More detailed work with roots will then be taken up in the next chapter.

Integer Exponents

Recall that use of a positive integer exponent is simply a shorthand for repeated multiplication. For example,

$$5^2 = 5 \cdot 5 \tag{1}$$

and

$$2^3 = 2 \cdot 2 \cdot 2. \tag{2}$$

In general, b^n stands for the quanitity b multiplied by itself n times. With this definition, the following *Laws of Exponents* hold.

Laws of Exponents

1. $b^r b^s = b^{r+s}$
2. $\dfrac{b^r}{b^s} = b^{r-s}$
3. $(b^r)^s = b^{rs}$

The Laws of Exponents are illustrated by the following examples.

Example 3.

a) $2^3 2^2 = (2 \cdot 2 \cdot 2)(2 \cdot 2) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 2^{3+2}$

b) $\dfrac{2^4}{2^2} = \dfrac{2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2} = \dfrac{2 \cdot 2 \cdot \cancel{2} \cdot \cancel{2}}{\cancel{2} \cdot \cancel{2}} = 2 \cdot 2 = 2^2 = 2^{4-2}$

c) $(2^3)^2 = (2^3)(2^3) = (2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6 = 2^{3 \cdot 2}$

Note that the second law only makes sense for $r > s$, since otherwise the exponent $r - s$ would be negative or 0. But actually, it turns out that we can create definitions for negative exponents and the 0 exponent, and consequently remove this restriction.

Negative exponents, as well as the 0 exponent, are simply defined in such a way that the Laws of Exponents will work for *all* integer exponents.

- For the 0 exponent, the first law implies that $b^0 b^1 = b^{0+1}$, and therefore $b^0 b = b$. If $b \neq 0$, we can divide both sides by b to obtain $b^0 = 1$ (there is one exception: 0^0 is not defined).
- For negative exponents, the second law implies that

$$b^{-n} = b^{0-n} = \frac{b^0}{b^n} = \frac{1}{b^n},$$

provided that $b \neq 0$. For example, $2^{-3} = 1/2^3 = 1/8$, and $2^{-4} = 1/2^4 = 1/16$.

Therefore, negative exponents and the 0 exponent are defined as follows:

Definition 4.

$$b^{-n} = \frac{1}{b^n} \qquad \text{and} \qquad b^0 = 1$$

provided that $b \neq 0$.

I **Example 5.** *Compute the exact values of* 4^{-3}, 6^0, *and* $\left(\frac{1}{5}\right)^{-2}$.

a) $4^{-3} = \dfrac{1}{4^3} = \dfrac{1}{64}$

b) $6^0 = 1$

c) $\left(\dfrac{1}{5}\right)^{-2} = \dfrac{1}{\left(\frac{1}{5}\right)^2} = \dfrac{1}{\frac{1}{25}} = 25$

We now have b^n defined for all integers n, in such a way that the Laws of Exponents hold. It may be surprising to learn that we can likewise define expressions using rational exponents, such as $2^{1/3}$, in a consistent manner. Before doing so, however, we'll need to take a detour and define *roots*.

Roots

Square Roots: Let's begin by defining the square root of a real number. We've used the square root in many sections in this text, so it should be a familiar concept. Nevertheless, in this section we'll look at square roots in more detail.

Definition 6. *Given a real number* a, *a "square root of* a*" is a number* x *such that* $x^2 = a$.

For example, 3 is a square root of 9 since $3^2 = 9$. Likewise, -4 is a square root of 16 since $(-4)^2 = 16$. In a sense, taking a square root is the "opposite" of squaring, so the definition of square root must be intimately connected with the graph of $y = x^2$, the squaring function. We investigate square roots in more detail by looking for solutions of the equation

$$x^2 = a. \tag{7}$$

There are three cases, each depending on the value and sign of a. In each case, the graph of the left-hand side of $x^2 = a$ is the parabola shown in **Figures 1**(a), (b), and (c).

- Case I: $a < 0$

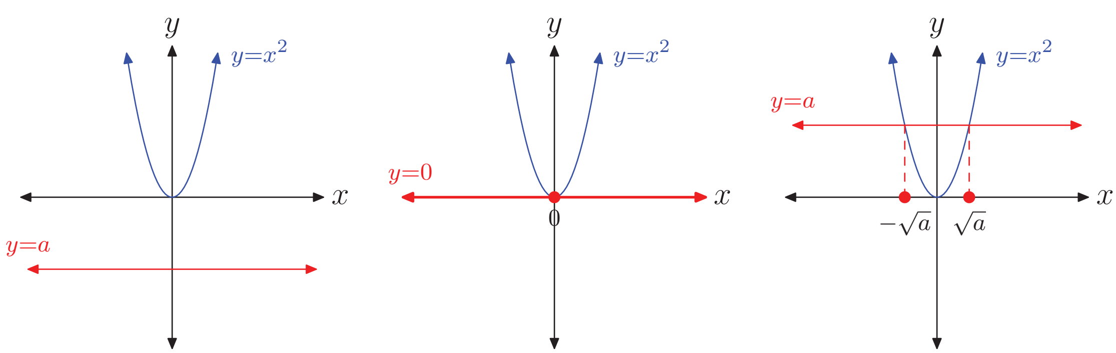
  The graph of the right-hand side of $x^2 = a$ is a horizontal line located a units *below* the x-axis. Hence, the graphs of $y = x^2$ and $y = a$ do not intersect and the equation $x^2 = a$ has no real solutions. This case is shown in **Figure 1**(a). It follows that a negative number has no square root.

- Case II: $a = 0$

  The graph of the right-hand side of $x^2 = 0$ is a horizontal line that coincides with the x-axis. The graph of $y = x^2$ intersects the graph of $y = 0$ at one point, at the vertex of the parabola. Thus, the only solution of $x^2 = 0$ is $x = 0$, as seen in **Figure 1**(b). The solution is the square root of 0, and is denoted $\sqrt{0}$, so it follows that $\sqrt{0} = 0$.

- Case III: $a > 0$

  The graph of the right-hand side of $x^2 = a$ is a horizontal line located a units *above* the x-axis. The graphs of $y = x^2$ and $y = a$ have two points of intersection, and therefore the equation $x^2 = a$ has two real solutions, as shown in **Figure 1**(c). The solutions of $x^2 = a$ are $x = \pm\sqrt{a}$. Note that we have two notations, one that calls for the positive solution and a second that calls for the negative solution.



(a) No real solutions. (b) One real solution. (c) Two real solutions.

Figure 1. The solutions of $x^2 = a$ depend upon the sign and value of a.

Let's look at some examples.

I **Example 8.** *What are the solutions of* $x^2 = -5$*?*

The graph of the left-hand side of $x^2 = -5$ is the parabola depicted in **Figure 1**(a). The graph of the right-hand side of $x^2 = -5$ is a horizontal line located 5 units below the x-axis. Thus, the graphs do not intersect and the equation $x^2 = -5$ has no real solutions.

You can also reason as follows. We're asked to find a solution of $x^2 = -5$, so you must find a number whose square equals -5. However, whenever you square a real number, the result is always nonnegative (zero or positive). It is not possible to square a real number and get -5.

Note that this also means that it is not possible to take the square root of a negative number. That is, $\sqrt{-5}$ is not a real number.

I **Example 9.** *What are the solutions of* $x^2 = 0$*?*

There is only one solution, namely $x = 0$. Note that this means that $\sqrt{0} = 0$.

I **Example 10.** *What are the solutions of* $x^2 = 25$*?*

The graph of the left-hand side of $x^2 = 25$ is the parabola depicted in **Figure 1**(c). The graph of the right-hand side of $x^2 = 25$ is a horizontal line located 25 units above the x-axis. The graphs will intersect in two points, so the equation $x^2 = 25$ has two real solutions.

The solutions of $x^2 = 25$ are called *square roots* of 25 and are written $x = \pm\sqrt{25}$. In this case, we can simplify further and write $x = \pm 5$.

It is extremely important to note the symmetry in **Figure 1**(c) and note that we have two real solutions, one negative and one positive. Thus, we need two notations, one for the positive square root of 25 and one for the negative square root 25.

Note that $(5)^2 = 25$, so $x = 5$ is the positive solution of $x^2 = 25$. For the positive solution, we use the notation

$$\sqrt{25} = 5.$$

This is pronounced "the positive square root of 25 is 5."

On the other hand, note that $(-5)^2 = 25$, so $x = -5$ is the negative solution of $x^2 = 25$. For the negative solution, we use the notation

$$-\sqrt{25} = -5.$$

This is pronounced "the negative square root of 25 is -5."

This discussion leads to the following detailed summary.

Summary: Square Roots

The solutions of $x^2 = a$ are called "square roots of a."

- Case I: $a < 0$. The equation $x^2 = a$ has no real solutions.
- Case II: $a = 0$. The equation $x^2 = a$ has one real solution, namely $x = 0$. Thus, $\sqrt{0} = 0$.
- Case III: $a > 0$. The equation $x^2 = a$ has two real solutions, $x = \pm\sqrt{a}$. The notation $\sqrt{a}$ calls for the positive square root of a, that is, the positive solution of $x^2 = a$. The notation $-\sqrt{a}$ calls for the negative square root of a, that is, the negative solution of $x^2 = a$.

Cube Roots: Let's move on to the definition of cube roots.

Definition 11. *Given a real number a, a "cube root of a" is a number x such that $x^3 = a$.*

For example, 2 is a cube root of 8 since $2^3 = 8$. Likewise, -4 is a cube root of -64 since $(-4)^3 = -64$. Thus, taking the cube root is the "opposite" of cubing, so the definition of cube root must be closely connected to the graph of $y = x^3$, the cubing function. Therefore, we look for solutions of

$$x^3 = a. \tag{12}$$

Because of the shape of the graph of $y = x^3$, there is only one case to consider. The graph of the left-hand side of $x^3 = a$ is shown in **Figure 2**. The graph of the right-hand side of $x^3 = a$ is a horizontal line, located a units above, on, or below the x-axis, depending on the sign and value of a. Regardless of the location of the horizontal line $y = a$, there will only be one point of intersection, as shown in **Figure 2**.

A detailed summary of cube roots follows.

Summary: Cube Roots

The solutions of $x^3 = a$ are called the "cube roots of a." Whether a is negative, zero, or positive makes no difference. There is exactly one real solution, namely $x = \sqrt[3]{a}$.

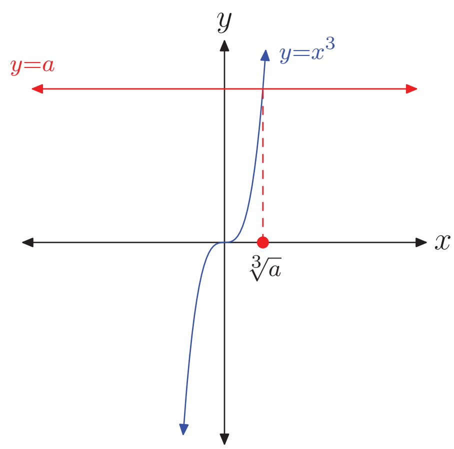


Figure 2. The graph of $y = x^3$ intersect the graph of $y = a$ in exactly one place.

Let's look at some examples.

I **Example 13.** *What are the solutions of $x^3 = 8$?*

The graph of the left-hand side of $x^3 = 8$ is the cubic polynomial shown in **Figure 2**. The graph of the right-hand side of $x^3 = 8$ is a horizontal line located 8 units above the x-axis. The graphs have one point of intersection, so the equation $x^3 = 8$ has exactly one real solution.[2]

The solutions of $x^3 = 8$ are called "cube roots of 8." As shown from the graph, there is exactly one real solution of $x^3 = 8$, namely $x = \sqrt[3]{8}$. Now since $(2)^3 = 8$, it follows that $x = 2$ is a real solution of $x^3 = 8$. Consequently, the cube root of 8 is 2, and we write

$$\sqrt[3]{8} = 2.$$

Note that in the case of cube root, there is no need for the two notations we saw in the square root case (one for the positive square root, one for the negative square root). This is because there is only one real cube root. Thus, the notation $\sqrt[3]{8}$ is pronounced "the cube root of 8."

I **Example 14.** *What are the solutions of $x^3 = 0$?*

There is only one solution of $x^3 = 0$, namely $x = 0$. This means that $\sqrt[3]{0} = 0$.

[2] There are also two other solutions, but they are both complex numbers, not real numbers. This textbook does not discuss complex numbers, but you may learn about them in more advanced courses.

I **Example 15.** *What are the solutions of $x^3 = -8$?*

The graph of the left-hand side of $x^3 = -8$ is the cubic polynomial shown in **Figure 2**. The graph of the right-hand side of $x^3 = -8$ is a horizontal line located 8 units below the x-axis. The graphs have only one point of intersection, so the equation $x^3 = -8$ has exactly one real solution, denoted $x = \sqrt[3]{-8}$. Now since $(-2)^3 = -8$, it follows that $x = -2$ is a real solution of $x^3 = -8$. Consequently, the cube root of -8 is -2, and we write

$$\sqrt[3]{-8} = -2.$$

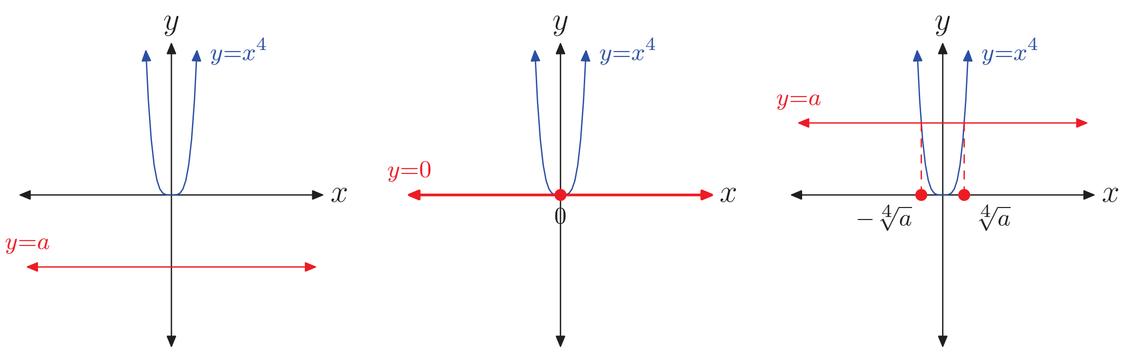
Again, because there is only one real solution of $x^3 = -8$, the notation $\sqrt[3]{-8}$ is pronounced "the cube root of -8." Note that, unlike the square root of a negative number, the cube root of a negative number is allowed.

Higher Roots: The previous discussions generalize easily to higher roots, such as fourth roots, fifth roots, sixth roots, etc.

Definition 16. *Given a real number a and a positive integer n, an "nth root of a" is a number x such that $x^n = a$.*

For example, 2 is a 6th root of 64 since $2^6 = 64$, and -3 is a fifth root of -243 since $(-3)^5 = -243$.

The case of *even* roots (i.e., when n is even) closely parallels the case of square roots. That's because when the exponent n is even, the graph of $y = x^n$ closely resembles that of $y = x^2$. For example, observe the case for fourth roots shown in **Figures 3**(a), (b), and (c).



(a) No real solutions. (b) One real solution. (c) Two real solutions.

Figure 3. The solutions of $x^4 = a$ depend upon the sign and value of a.

The discussion for *even* nth roots closely parallels that presented in the introduction of square roots, so without further ado, we go straight to the summary.

Summary: Even nth Roots

If n is a positive even integer, then the solutions of $x^n = a$ are called "nth roots of a."

- Case I: $a < 0$. The equation $x^n = a$ has no real solutions.
- Case II: $a = 0$. The equation $x^n = a$ has exactly one real solution, namely $x = 0$. Thus, $\sqrt[n]{0} = 0$.
- Case III: $a > 0$. The equation $x^n = a$ has two real solutions, $x = \pm\sqrt[n]{a}$. The notation $\sqrt[n]{a}$ calls for the positive nth root of a, that is, the positive solution of $x^n = a$. The notation $-\sqrt[n]{a}$ calls for the negative nth root of a, that is, the negative solution of $x^n = a$.

Likewise, the case of *odd* roots (i.e., when n is odd) closely parallels the case of cube roots. That's because when the exponent n is odd, the graph of $y = x^n$ closely resembles that of $y = x^3$. For example, observe the case for fifth roots shown in **Figure 4**.

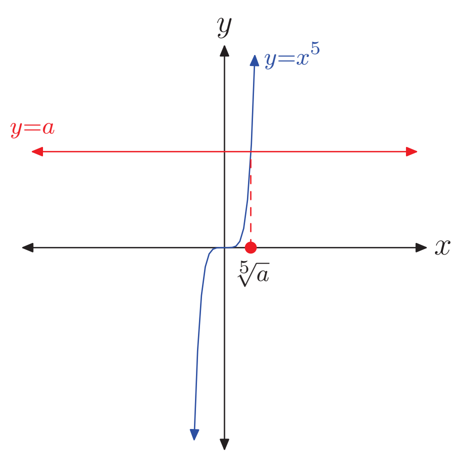


Figure 4. The graph of $y = x^5$ intersects the graph of $y = a$ in exactly one place.

The discussion of *odd* nth roots closely parallels the introduction of cube roots which we discussed earlier. So, without further ado, we proceed straight to the summary.

Summary: Odd nth Roots

If n is a positive odd integer, then the solutions of $x^n = a$ are called the "nth roots of a." Whether a is negative, zero, or positive makes no difference. There is exactly one real solution of $x^n = a$, denoted $x = \sqrt[n]{a}$.

Remark 17. The symbols $\sqrt{}$ and $\sqrt[n]{}$ for square root and nth root, respectively, are also called *radicals*.

We'll close this section with a few more examples.

I **Example 18.** *What are the solutions of $x^4 = 16$?*

The graph of the left-hand side of $x^4 = 16$ is the quartic polynomial shown in **Figure 3**(c). The graph of the right-hand side of $x^4 = 16$ is a horizontal line, located 16 units above the x-axis. The graphs will intersect in two points, so the equation $x^4 = 16$ has two real solutions.

The solutions of $x^4 = 16$ are called *fourth roots* of 16 and are written $x = \pm\sqrt[4]{16}$. It is extremely important to note the symmetry in **Figure 3**(c) and note that we have two real solutions of $x^4 = 16$, one of which is negative and the other positive. Hence, we need two notations, one for the positive fourth root of 16 and one for the negative fourth root of 16.

Note that $2^4 = 16$, so $x = 2$ is the positive real solution of $x^4 = 16$. For this positive solution, we use the notation

$$\sqrt[4]{16} = 2.$$

This is pronounced "the positive fourth root of 16 is 2."

On the other hand, note that $(-2)^4 = 16$, so $x = -2$ is the negative real solution of $x^4 = 16$. For this negative solution, we use the notation

$$-\sqrt[4]{16} = -2. \tag{19}$$

This is pronounced "the negative fourth root of 16 is -2."

I **Example 20.** *What are the solutions of $x^5 = -32$?*

The graph of the left-hand side of $x^5 = -32$ is the quintic polynomial pictured in **Figure 4**. The graph of the right-hand side of $x^5 = -32$ is a horizontal line, located 32 units below the x-axis. The graphs have one point of intersection, so the equation $x^5 = -32$ has exactly one real solution.

The solutions of $x^5 = -32$ are called "fifth roots of -32." As shown from the graph, there is exactly one real solution of $x^5 = -32$, namely $x = \sqrt[5]{-32}$. Now since $(-2)^5 = -32$, it follows that $x = -2$ is a solution of $x^5 = -32$. Consequently, the fifth root of -32 is -2, and we write

$$\sqrt[5]{-32} = -2.$$

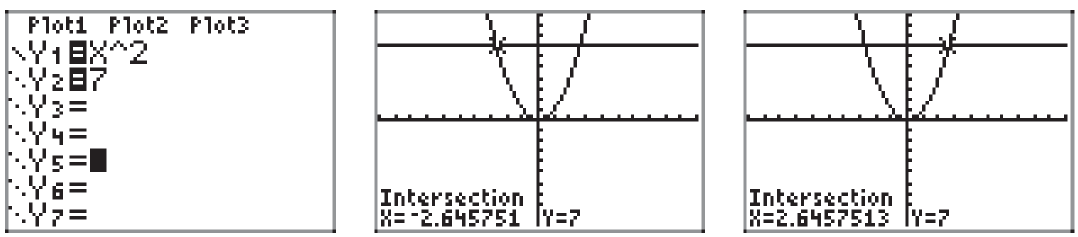
Because there is only one real solution, the notation $\sqrt[5]{-32}$ is pronounced "the fifth root of -32." Again, unlike the square root or fourth root of a negative number, the fifth root of a negative number is allowed.

Not all roots simplify to rational numbers. If that were the case, it would not even be necessary to implement radical notation. Consider the following example.

I **Example 21.** *Find all real solutions of the equation $x^2 = 7$, both graphically and algebraically, and compare your results.*

We could easily sketch rough graphs of $y = x^2$ and $y = 7$ by hand, but let's seek a higher level of accuracy by asking the graphing calculator to handle this task.

- Load the equation $y = x^2$ and $y = 7$ into `Y1` and `Y2` in the calculator's `Y=` menu, respectively. This is shown in **Figure 5**(a).
- Use the `intersect` utility on the graphing calculator to find the coordinates of the points of intersection. The x-coordinates of these points, shown in **Figure 5**(b) and (c), are the solutions to the equation $x^2 = 7$.



(a) (b) (c)

Figure 5. The solutions of $x^2 = 7$ are $x \approx -2.645751$ or $x \approx 2.6457513$.

Guidelines for Reporting Graphing Calculator Solutions. Recall the standard method for reporting graphing calculator results on your homework:

- Copy the image from your viewing window onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax, then label each graph with its equation, as shown in **Figure 6**.
- Drop dashed vertical lines from each point of intersection to the x-axis. Shade and label your solutions on the x-axis.

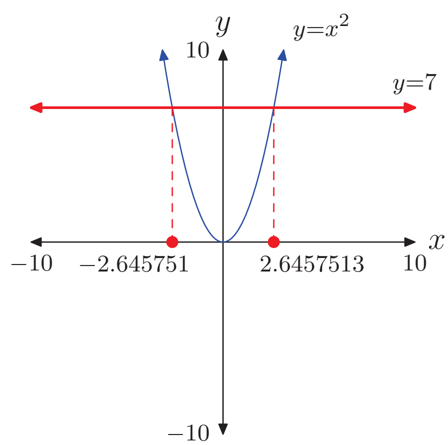


Figure 6. The solutions of $x^2 = 7$ are $x \approx -2.645751$ or $x \approx 2.6457513$.

Hence, the **approximate** solutions are $x \approx -2.645751$ or $x \approx 2.6457513$.

On the other hand, to find analytic solutions of $x^2 = 7$, we simply take plus or minus the square root of 7.

$$x^2 = 7$$
$$x = \pm\sqrt{7}$$

To compare these **exact** solutions with the approximate solutions found by using the graphing calculator, use a calculator to compute $\pm\sqrt{7}$, as shown in **Figure 7**.

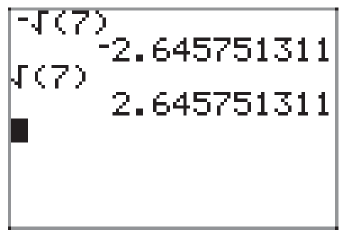


Figure 7. Approximating $\pm\sqrt{7}$.

Note that these approximations of $-\sqrt{7}$ and $\sqrt{7}$ agree quite nicely with the solutions found using the graphing calculator's `intersect` utility and reported in **Figure 6**.

Both $-\sqrt{7}$ and $\sqrt{7}$ are examples of *irrational* numbers, that is, numbers that cannot be expressed in the form p/q, where p and q are integers.

Rational Exponents

As with the definition of negative and zero exponents, discussed earlier in this section, it turns out that rational exponents can be defined in such a way that the Laws of Exponents will still apply (and in fact, there's only one way to do it).

The third law gives us a hint on how to define rational exponents. For example, suppose that we want to define $2^{1/3}$. Then by the third law,

$$\left(2^{\frac{1}{3}}\right)^3 = 2^{\frac{1}{3}\cdot 3} = 2^1 = 2,$$

so, by taking cube roots of both sides, we must define $2^{1/3}$ by the formula[3]

$$2^{\frac{1}{3}} = \sqrt[3]{2}.$$

The same argument shows that if n is any odd positive integer, then $2^{1/n}$ must be defined by the formula

$$2^{\frac{1}{n}} = \sqrt[n]{2}.$$

However, for an even integer n, there appears to be a choice. Suppose that we want to define $2^{1/2}$. Then

[3] Recall that the equation $x^3 = a$ has a unique solution $x = \sqrt[3]{a}$.

$$\left(2^{\frac{1}{2}}\right)^2 = 2^{\frac{1}{2}\cdot 2} = 2^1 = 2,$$

so

$$2^{\frac{1}{2}} = \pm\sqrt{2}.$$

However, the negative choice for the exponent $1/2$ leads to problems, because then certain expressions are not defined. For example, it would follow from the third law that

$$(2^{\frac{1}{2}})^{\frac{1}{2}} = -\sqrt{-\sqrt{2}}.$$

But $-\sqrt{2}$ is negative, so $\sqrt{-\sqrt{2}}$ is not defined. Therefore, it only makes sense to use the positive choice. Thus, for all n, even and odd, $2^{1/n}$ is defined by the formula

$$2^{\frac{1}{n}} = \sqrt[n]{2}.$$

In a similar manner, for a general positive rational $\frac{m}{n}$, the third law implies that

$$2^{\frac{m}{n}} = (2^m)^{\frac{1}{n}} = \sqrt[n]{2^m}.$$

But also,

$$2^{\frac{m}{n}} = (2^{\frac{1}{n}})^m = (\sqrt[n]{2})^m.$$

Thus,

$$2^{\frac{m}{n}} = \sqrt[n]{2^m} = (\sqrt[n]{2})^m.$$

Finally, negative rational exponents are defined in the usual manner for negative exponents:

$$2^{-\frac{m}{n}} = \frac{1}{2^{\frac{m}{n}}}$$

More generally, here is the final general definition. With this definition, the Laws of Exponents hold for all rational exponents.

Definition 22. *For a positive rational exponent $\frac{m}{n}$, and $b > 0$,*

$$b^{\frac{m}{n}} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m. \tag{23}$$

For a negative rational exponent $-\frac{m}{n}$,

$$b^{-\frac{m}{n}} = \frac{1}{b^{\frac{m}{n}}}. \tag{24}$$

Remark 25. For $b < 0$, the same definitions make sense only when n is odd. For example $(-2)^{\frac{1}{4}}$ is not defined.

I **Example 26.** *Compute the exact values of* (a) $4^{\frac{5}{2}}$, (b) $64^{\frac{2}{3}}$, *and* (c) $81^{-\frac{3}{4}}$.

a) $4^{\frac{5}{2}} = \left(4^{\frac{1}{2}}\right)^5 = (\sqrt{4})^5 = 2^5 = 32$

b) $64^{\frac{2}{3}} = \left(64^{\frac{1}{3}}\right)^2 = (\sqrt[3]{64})^2 = 4^2 = 16$

c) $81^{-\frac{3}{4}} = \dfrac{1}{81^{\frac{3}{4}}} = \dfrac{1}{\left(81^{\frac{1}{4}}\right)^3} = \dfrac{1}{(\sqrt[4]{81})^3} = \dfrac{1}{3^3} = \dfrac{1}{27}$

I **Example 27.** *Simplify the following expressions, and write them in the form* x^r*:*

$$\text{a) } x^{\frac{2}{3}}x^{\frac{1}{4}}, \qquad \text{b) } \frac{x^{\frac{2}{3}}}{x^{\frac{1}{4}}}, \qquad \text{c) } \left(x^{-\frac{2}{3}}\right)^{\frac{1}{4}}$$

a) $x^{\frac{2}{3}}x^{\frac{1}{4}} = x^{\frac{2}{3}+\frac{1}{4}} = x^{\frac{8}{12}+\frac{3}{12}} = x^{\frac{11}{12}}$

b) $\dfrac{x^{\frac{2}{3}}}{x^{\frac{1}{4}}} = x^{\frac{2}{3}-\frac{1}{4}} = x^{\frac{8}{12}-\frac{3}{12}} = x^{\frac{5}{12}}$

c) $\left(x^{-\frac{2}{3}}\right)^{\frac{1}{4}} = x^{-\frac{2}{3}\cdot\frac{1}{4}} = x^{-\frac{2}{12}} = x^{-\frac{1}{6}}$

I **Example 28.** *Use rational exponents to simplify* $\sqrt[5]{\sqrt{x}}$*, and write it as a single radical.*

$$\sqrt[5]{\sqrt{x}} = (\sqrt{x})^{\frac{1}{5}} = \left(x^{\frac{1}{2}}\right)^{\frac{1}{5}} = x^{\frac{1}{2}\cdot\frac{1}{5}} = x^{\frac{1}{10}} = \sqrt[10]{x}$$

I **Example 29.** *Use a calculator to approximate* $2^{5/8}$.

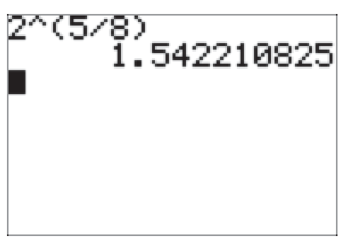


Figure 8. $2^{5/8} \approx 1.542210825$

Irrational Exponents

What about irrational exponents? Is there a way to define numbers like $2^{\sqrt{2}}$ and 3^{π}? It turns out that the answer is yes. While a rigorous definition of b^s when s is irrational is beyond the scope of this book, it's not hard to see how one could proceed to find a value for such a number. For example, if we want to compute the value of $2^{\sqrt{2}}$, we can start with rational approximations for $\sqrt{2}$. Since $\sqrt{2} = 1.41421356237310\ldots$, the successive powers

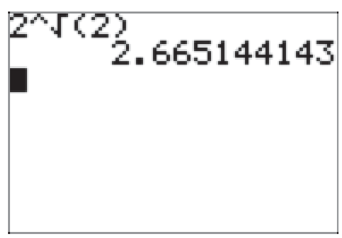
$$2^1,\ 2^{1.4},\ 2^{1.41},\ 2^{1.414},\ 2^{1.4142},\ 2^{1.41421},\ 2^{1.414213},$$
$$2^{1.4142135},\ 2^{1.41421356},\ 2^{1.414213562},\ 2^{1.4142135623},\ldots$$

should be closer and closer approximations to the desired value of $2^{\sqrt{2}}$.

In fact, using more advanced mathematical theory (ultimately based on the actual construction of the real number system), it can be shown that these powers approach a single real number, and we define $2^{\sqrt{2}}$ to be that number. Using your calculator, you can observe this convergence and obtain an approximation by computing the powers above.

t	$f(t) = 2^t$
1	2
1.4	2.639015822
1.41	2.657371628
1.414	2.664749650
1.4142	2.665119089
1.41421	2.665137562
1.414213	2.665143104
1.4142135	2.665144027
1.41421356	2.665144138
1.414213562	2.665144142
1.4142135623	2.665144143

(a) Approximations of $2^{\sqrt{2}}$



(b) $2^{\sqrt{2}} \approx 2.665144143$

Figure 9.

The last value in the table in **Figure 9**(a) is a correct approximation of $2^{\sqrt{2}}$ to 10 digits of accuracy. Your calculator will obtain this same approximation when you ask it to compute $2^{\sqrt{2}}$ directly (see **Figure 9**(b)).

In a similar manner, b^s can be defined for any irrational exponent s and any $b > 0$. Combined with the earlier work in this section, it follows that b^s is defined for every real exponent s.

4.2 *Exercises*

In **Exercises 1-12**, compute the exact value.

1. 3^{-5}

2. 4^2

3. $(3/2)^3$

4. $(2/3)^1$

5. 6^{-2}

6. 4^{-3}

7. $(2/3)^{-3}$

8. $(1/3)^{-3}$

9. 7^1

10. $(3/2)^{-4}$

11. $(5/6)^3$

12. 3^2

In **Exercises 13-24**, perform each of the following tasks for the given equation.

i. Load the left- and right-hand sides of the given equation into `Y1` and `Y2`, respectively. Adjust the `WINDOW` parameters until all points of intersection (if any) are visible in your viewing window. Use the `intersect` utility in the `CALC` menu to determine the coordinates of any points of intersection.

ii. Make a copy of the image in your viewing window on your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Label each graph with its equation. Drop dashed vertical lines from each point of intersection to the x-axis, then shade and label each solution of the given equation on the x-axis. *Remember to draw all lines with a ruler.*

iii. Solve each problem algebraically. Use a calculator to approximate any radicals and compare these solutions with those found in parts (i) and (ii).

13. $x^2 = 5$

14. $x^2 = 7$

15. $x^2 = -7$

16. $x^2 = -3$

17. $x^3 = -6$

18. $x^3 = -4$

19. $x^4 = 4$

20. $x^4 = -7$

21. $x^5 = 8$

22. $x^5 = 4$

23. $x^6 = -5$

24. $x^6 = 9$

In **Exercises 25-40**, simplify the given radical expression.

25. $\sqrt{49}$

26. $\sqrt{121}$

27. $\sqrt{-36}$

28. $\sqrt{-100}$

29. $\sqrt[3]{27}$

30. $\sqrt[3]{-1}$

31. $\sqrt[3]{-125}$

32. $\sqrt[3]{64}$

33. $\sqrt[4]{-16}$

34. $\sqrt[4]{81}$

35. $\sqrt[4]{16}$

36. $\sqrt[4]{-625}$

37. $\sqrt[5]{-32}$

38. $\sqrt[5]{243}$

39. $\sqrt[5]{1024}$

40. $\sqrt[5]{-3125}$

41. Compare and contrast $\sqrt{(-2)^2}$ and $(\sqrt{-2})^2$.

42. Compare and contrast $\sqrt[4]{(-3)^4}$ and $(\sqrt[4]{-3})^4$.

43. Compare and contrast $\sqrt[3]{(-5)^3}$ and $(\sqrt[3]{-5})^3$.

44. Compare and contrast $\sqrt[5]{(-2)^5}$ and $(\sqrt[5]{-2})^5$.

In **Exercises 45-56**, compute the exact value.

45. $25^{-\frac{3}{2}}$

46. $16^{-\frac{5}{4}}$

47. $8^{\frac{4}{3}}$

48. $625^{-\frac{3}{4}}$

49. $16^{\frac{3}{2}}$

50. $64^{\frac{2}{3}}$

51. $27^{\frac{2}{3}}$

52. $625^{\frac{3}{4}}$

53. $256^{\frac{5}{4}}$

54. $4^{-\frac{3}{2}}$

55. $256^{-\frac{3}{4}}$

56. $81^{-\frac{5}{4}}$

In **Exercises 57-64**, simplify the product, and write your answer in the form x^r.

57. $x^{\frac{5}{4}}x^{\frac{5}{4}}$

58. $x^{\frac{5}{3}}x^{-\frac{5}{4}}$

59. $x^{-\frac{1}{3}}x^{\frac{5}{2}}$

60. $x^{-\frac{3}{5}}x^{\frac{3}{2}}$

61. $x^{\frac{4}{5}}x^{-\frac{4}{3}}$

62. $x^{-\frac{5}{4}}x^{\frac{1}{2}}$

63. $x^{-\frac{2}{5}}x^{-\frac{3}{2}}$

64. $x^{-\frac{5}{4}}x^{\frac{5}{2}}$

In **Exercises 65-72**, simplify the quotient, and write your answer in the form x^r.

65. $\dfrac{x^{-\frac{5}{4}}}{x^{\frac{1}{5}}}$

66. $\dfrac{x^{-\frac{2}{3}}}{x^{\frac{1}{4}}}$

67. $\dfrac{x^{-\frac{1}{2}}}{x^{-\frac{3}{5}}}$

68. $\dfrac{x^{-\frac{5}{2}}}{x^{\frac{2}{5}}}$

69. $\dfrac{x^{\frac{3}{5}}}{x^{-\frac{1}{4}}}$

70. $\dfrac{x^{\frac{1}{3}}}{x^{-\frac{1}{2}}}$

71. $\dfrac{x^{-\frac{5}{4}}}{x^{\frac{2}{3}}}$

72. $\dfrac{x^{\frac{1}{3}}}{x^{\frac{1}{2}}}$

In **Exercises 73-80**, simplify the expression, and write your answer in the form x^r.

73. $\left(x^{\frac{1}{2}}\right)^{\frac{4}{3}}$

74. $\left(x^{-\frac{1}{2}}\right)^{-\frac{1}{2}}$

75. $\left(x^{-\frac{5}{4}}\right)^{\frac{1}{2}}$

76. $\left(x^{-\frac{1}{5}}\right)^{-\frac{3}{2}}$

77. $\left(x^{-\frac{1}{2}}\right)^{\frac{3}{2}}$

78. $\left(x^{-\frac{1}{3}}\right)^{-\frac{1}{2}}$

79. $\left(x^{\frac{1}{5}}\right)^{-\frac{1}{2}}$

80. $\left(x^{\frac{2}{5}}\right)^{-\frac{1}{5}}$

4.2 *Answers*

1. $\dfrac{1}{243}$

3. $\dfrac{27}{8}$

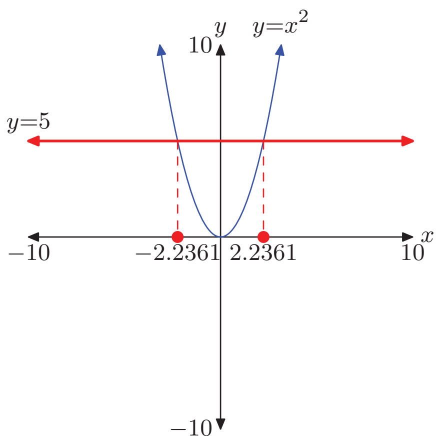
5. $\dfrac{1}{36}$

7. $\dfrac{27}{8}$

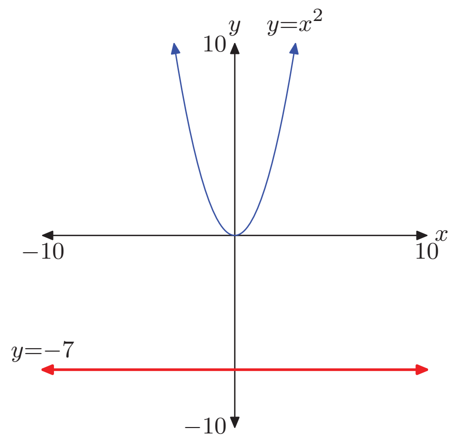
9. 7

11. $\dfrac{125}{216}$

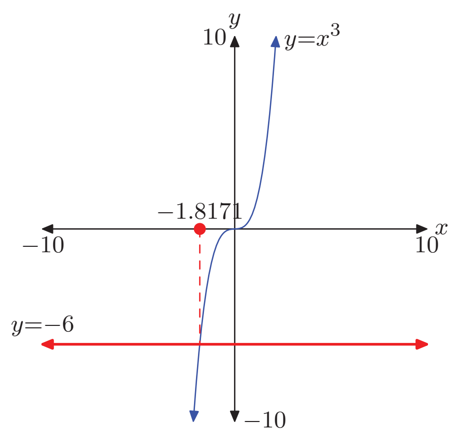
13. Solutions: $x = \pm\sqrt{5}$



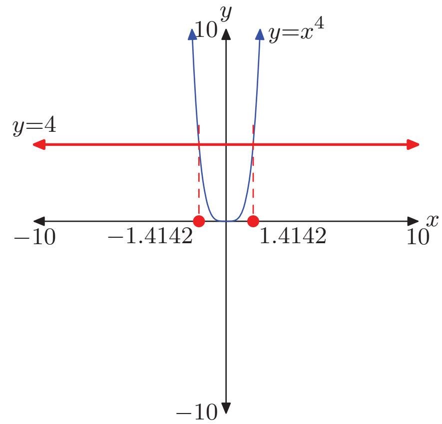
15. No real solutions.



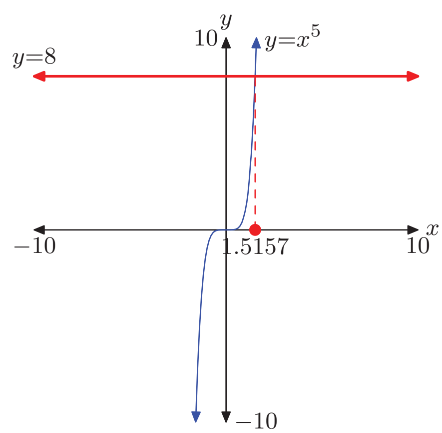
17. $x = \sqrt[3]{-6}$



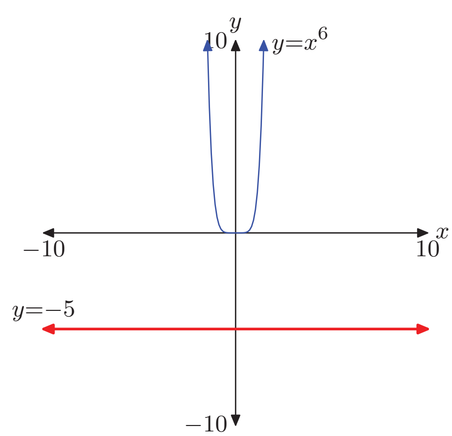
19. Solutions: $x = \pm\sqrt[4]{4}$



21. $x = \sqrt[5]{8}$



23. No real solutions.



25. 7

27. Not a real number.

29. 3

31. -5

33. Not a real number.

35. 2

37. -2

39. 4

41. $\sqrt{(-2)^2} = 2$, while $(\sqrt{-2})^2$ is not a real number.

43. Both equal -5.

45. $\dfrac{1}{125}$

47. 16

49. 64

51. 9

53. 1024

55. $\dfrac{1}{64}$

57. $x^{\frac{5}{2}}$

59. $x^{\frac{13}{6}}$

61. $x^{-\frac{8}{15}}$

63. $x^{-\frac{19}{10}}$

65. $x^{-\frac{29}{20}}$

67. $x^{\frac{1}{10}}$

69. $x^{\frac{17}{20}}$

71. $x^{-\frac{23}{12}}$

73. $x^{\frac{2}{3}}$

75. $x^{-\frac{5}{8}}$

77. $x^{-\frac{3}{4}}$

79. $x^{-\frac{1}{10}}$

4.3 Exponential Functions

Let's suppose that the current population of the city of Pleasantville is 10 000 and that the population is growing at a rate of 2% per year. In order to analyze the population growth over a period of years, we'll try to develop a formula for the population as a function of time, and then graph the result.

First, note that at the end of one year, the population increase is 2% of 10 000, or 200 people. We would now have 10 200 people in Pleasantville. At the end of the second year, take another 2% of 10 200, which is an increase of 204 people, for a total of 10 404. Because the increase each year is not constant, the graph of population versus time cannot be a line. Hence, our eventual population function will not be linear.

To develop our population formula, we start by letting the function $P(t)$ represent the population of Pleasantville at time t, where we measure t in years. We will start time at $t = 0$ when the initial population of Pleasantville is 10 000. In other words, $P(0) = 10\,000$. The key to understanding this example is the fact that the population increases by 2% each year. We are making an assumption here that this overall growth accounts for births, deaths, and people coming into and leaving Pleasantville. That is, at the end of the first year, the population of Pleasantville will be 102% of the initial population. Thus,

$$P(1) = 1.02P(0) = 1.02(10\,000). \tag{1}$$

We could multiply out the right side of this equation, but it will actually be more useful to leave it in its current form.

Now each year the population increases by 2%. Therefore, at the end of the second year, the population will be 102% of the population at the end of the first year. In other words,

$$P(2) = 1.02P(1). \tag{2}$$

If we replace $P(1)$ in **equation (2)** with the result found in **equation (1)**, then

$$P(2) = (1.02)(1.02)(10\,000) = (1.02)^2(10\,000). \tag{3}$$

Let's iterate one more year. At the end of the third year, the population will be 102% of the population at the end of the second year, so

$$P(3) = 1.02P(2). \tag{4}$$

However, if we replace $P(2)$ in **equation (4)** with the result found in **equation (3)**, we obtain

$$P(3) = (1.02)(1.02)^2(10\,000) = (1.02)^3(10\,000). \tag{5}$$

The pattern should now be clear. The population at the end of t years is given by the function

$$P(t) = (1.02)^t(10\,000).$$

It is traditional in mathematics and science to place the initial population in front in this formula, writing instead

$$P(t) = 10\,000(1.02)^t. \tag{6}$$

Our function $P(t)$ is defined by **equation (6)** for all positive integers $\{1, 2, 3, \ldots\}$, and $P(0) = 10\,000$, the initial population. **Figure 1** shows a plot of our function. Although points are plotted only at integer values of t from 0 to 40, that's enough to show the trend of the population over time. The population starts at 10 000, increases over time, and the yearly increase (the difference in population from one year to the next) also gets larger as time passes.

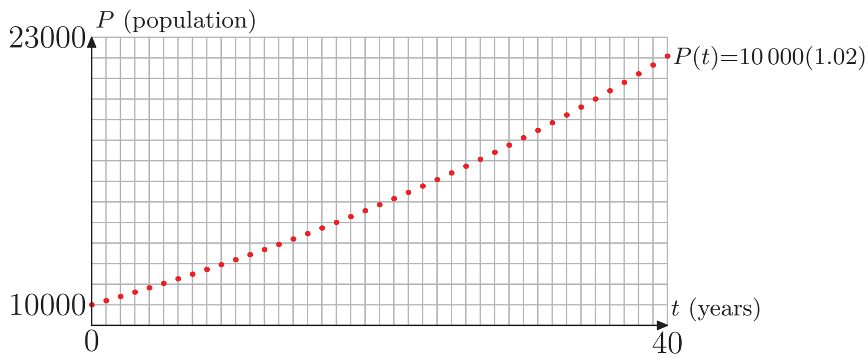


Figure 1. Graph of population $P(t)$ of Pleasantville for $t = 0, 1, 2, 3, \ldots$

I **Example 7.** *We can now use the function $P(t)$ to predict the population in later years. Assuming that the growth rate of 2% continues, what will the population of Pleasantville be after 40 years? What will it be after 100 years?*

Substitute $t = 40$ and $t = 100$ into **equation (6)**. The population in 40 years will be

$$P(40) = 10\,000(1.02)^{40} \approx 22\,080,$$

and the population in 100 years will be

$$P(100) = 10\,000(1.02)^{100} \approx 72\,446.$$

What would be different if we had started with a population of 12 000? By tracing over our previous steps, it should be easy to see that the new formula would be

$$P(t) = 12\,000(1.02)^t.$$

Similarly, if the growth rate had been 3% per year instead of 2%, then we would have ended up with the formula

$$P(t) = 10\,000(1.03)^t.$$

Thus, by letting P_0 represent the initial population, and r represent the growth rate (in decimal form), we can generalize the formula to

$$P(t) = P_0(1+r)^t. \tag{8}$$

Note that our formula for the function $P(t)$ is different from the previous functions that we've studied so far, in that the input variable t is part of the exponent in the formula. Thus, this is a new type of function.

Now let's contrast the situation in Pleasantville with the population dynamics of Ghosttown. Ghosttown also starts with a population of 10 000, but several factories have closed, so some people are leaving for better opportunities. In this case, the population of Ghosttown is *decreasing* at a rate of 2% per year. We'll again develop a formula for the population as a function of time, and then graph the result.

First, note that at the end of one year, the population decrease is 2% of 10 000, or 200 people. We would now have 9 800 people left in Ghosttown. At the end of the second year, take another 2% of 9 800, which is a decrease of 196 people, for a total of 9 604. As before, because the decrease each year is not constant, the graph of population versus time cannot be a line, so our eventual population function will not be linear.

Now let the function $P(t)$ represent the population of Ghosttown at time t, where we measure t in years. The initial population of Ghosttown at $t = 0$ is 10 000, so $P(0) = 10\,000$. Since the population decreases by 2% each year, at the end of the first year the population of Ghosttown will be 98% of the initial population. Thus,

$$P(1) = 0.98P(0) = 0.98(10\,000). \tag{9}$$

Each year the population deccreases by 2%. Therefore, at the end of the second year, the population will be 98% of the population at the end of the first year. In other words,

$$P(2) = 0.98P(1). \tag{10}$$

If we replace $P(1)$ in **equation (10)** with the result found in **equation (9)**, then

$$P(2) = (0.98)(0.98)(10\,000) = (0.98)^2(10\,000). \tag{11}$$

Let's iterate one more year. At the end of the third year, the population will be 98% of the population at the end of the second year, so

$$P(3) = 0.98P(2). \tag{12}$$

However, if we replace $P(2)$ in **equation (12)** with the result found in **equation (11)**, we obtain

$$P(3) = (0.98)(0.98)^2(10\,000) = (0.98)^3(10\,000). \tag{13}$$

The pattern should now be clear. The population at the end of t years is given by the function

$$P(t) = (0.98)^t(10\,000),$$

or equivalently,

$$P(t) = 10\,000(0.98)^t. \tag{14}$$

Our function $P(t)$ is defined by **equation (14)** for all positive integers $\{1, 2, 3, \ldots\}$, and $P(0) = 10\,000$, the initial population. **Figure 2** shows a plot of our function. Although points are plotted only at integer values of t from 0 to 40, that's enough to show the trend of the population over time. The population starts at 10 000, decreases over time, and the yearly decrease (the difference in population from one year to the next) also gets smaller as time passes.

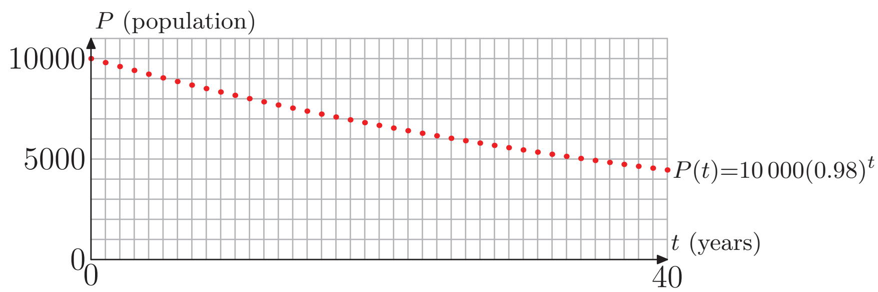


Figure 2. Graph of population $P(t)$ of Ghosttown for $t = 0, 1, 2, 3, \ldots$

I **Example 15.** *Assuming that the rate of decrease continues at 2%, predict the population of Ghosttown after 40 years and after 100 years.*

Substitute $t = 40$ and $t = 100$ into **equation (14)**. The population in 40 years will be

$$P(40) = 10\,000(0.98)^{40} \approx 4457,$$

and the population in 100 years will be

$$P(100) = 10\,000(0.98)^{100} \approx 1326.$$

Note that if we had instead started with a population of 9 000, for example, then the new formula would be

$$P(t) = 9\,000(0.98)^t.$$

Similarly, if the rate of decrease had been 5% per year instead of 2%, then we would have ended up with the formula

$$P(t) = 10\,000(0.95)^t.$$

Thus, by letting P_0 represent the initial population, and r represent the growth rate (in decimal form), we can generalize the formula to

$$P(t) = P_0(1 - r)^t. \tag{16}$$

Definition

As noted before, our functions $P(t)$ in our Pleasantville and Ghosttown examples are a new type of function, because the input variable t is part of the exponent in the formula.

Definition 17. *An* exponential function *is a function of the form*

$$f(t) = b^t,$$

where $b > 0$ *and* $b \neq 1$. b *is called the* **base** *of the exponential function.*

More generally, a function of the form

$$f(t) = Ab^t,$$

where $b > 0$, $b \neq 1$, *and* $A \neq 0$, *is also referred to as an exponential function. In this case, the value of the function when* $t = 0$ *is* $f(0) = A$, *so* A *is the* **initial amount***.*

In applications, you will almost always encounter exponential functions in the more general form Ab^t. In fact, note that in the previous population examples, the function $P(t)$ has this form $P(t) = Ab^t$, with $A = P_0$, $b = 1 + r$ in Pleasantville, and $b = 1 - r$ in Ghosttown. In particular, $A = P_0$ is the initial population.

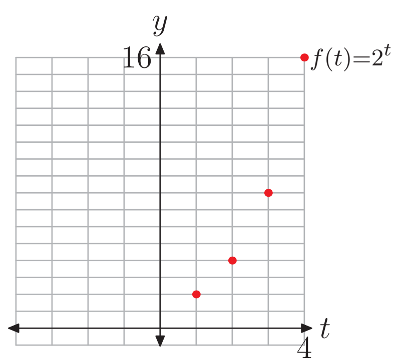
Since exponential functions are often used to model processes that vary with time, we usually use the input variable t (although of course any variable can be used). Also, you may be curious why the definition says $b \neq 1$, since 1^t just equals 1. We'll explain this curiosity at the end of this section.

Graphs of Exponential Functions

We'll develop the properties for the basic exponential function b^t first, and then note the minor changes for the more general form Ab^t. For a working example, let's use base $b = 2$, and let's compute some values of $f(t) = 2^t$ and plot the result (see **Figure 3**).

t	$f(t) = 2^t$
1	2
2	4
3	8
4	16

(a)



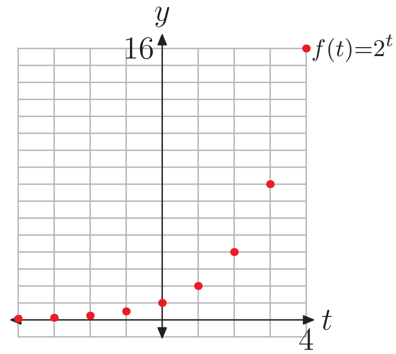
(b)

Figure 3. Plotting points $(t, f(t))$ defined by the function $f(t) = 2^t$, with $t = 1, 2, 3, 4, \ldots$.

Recall from the previous section that 2^t is also defined for negative exponents t and the 0 exponent. Thus, the exponential function $f(t) = 2^t$ is defined for all integers. **Figure 4** shows a new table and plot with points added at 0 and negative integer values.

t	$f(t) = 2^t$
-4	$1/16$
-3	$1/8$
-2	$1/4$
-1	$1/2$
0	0
1	2
2	4
3	8
4	16

(a)



(b)

Figure 4. Plotting points $(t, f(t))$ defined by the function $f(t) = 2^t$, with $t = \ldots, -3, -2, -1, 0, 1, 2, 3 \ldots$.

However, the previous section showed that 2^t is also defined for rational and irrational exponents. Therefore, the domain of the exponential function $f(t) = 2^t$ is the set of all real numbers. When we add in the values of the function at all rational and irrational values of t, we obtain a final continuous curve as shown in **Figure 5**.

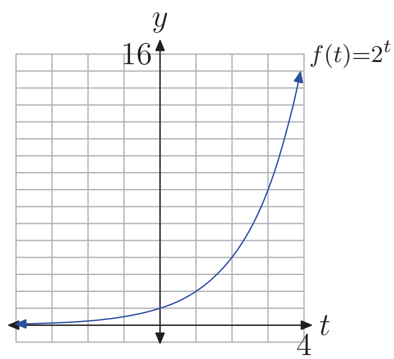


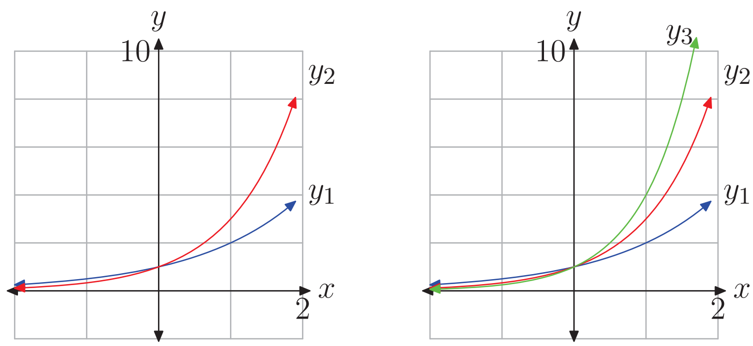
Figure 5.

Note several properties of the graph in **Figure 5**:

a) Moving from left to right, the curve rises, which means that the function increases as t increases. In fact, the function increases rapidly for positive t.
b) The graph lies above the t-axis, so the values of the function are always positive. Therefore, the range of the function is $(0, \infty)$.
c) The graph has a horizontal asymptote $y = 0$ (the t-axis) on the left side. This means that the function almost "dies out" (the values get closer and closer to 0) as t approaches $-\infty$.

What about the graphs of other exponential functions with different bases? We'll use the calculator to explore several of these.

First, use your calculator to compare $y_1(x) = 2^x$ and $y_2(x) = 3^x$. As can be seen in **Figure 6**(a), the graph of 3^x rises faster that 2^x for $x > 0$, and dies out faster for $x < 0$.

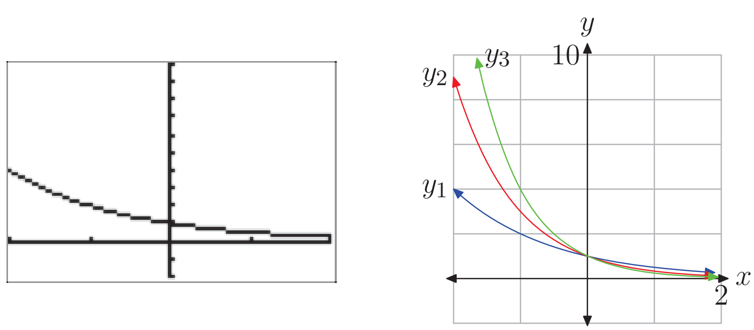


(a) (b)

Figure 6. Comparing functions $y_1(x) = 2^x$, $y_2(x) = 3^x$, and $y_3(x) = 4^x$

Next, add in $y_3(x) = 4^x$. The result is shown in **Figure 6**(b). Again, increasing the size of the base to $b = 4$ results in a function which rises even faster on the right and likewise dies out faster on the left. If you continue to increase the size of the base b, you'll see that this trend continues. That's not terribly surprising because, if we compute the value of these functions at a fixed positive x, for example at $x = 2$, then the values increase: $2^2 < 3^2 < 4^2 < \ldots$. Similarly, at $x = -2$, the values decrease: $2^{-2} > 3^{-2} > 4^{-2} > \ldots$.

All of the functions in our experiments so far share the properties listed in (a)–(c) above: the function increases, the range is $(0, \infty)$, and the graph has a horizontal asymptote $y = 0$ on the left side. Now let's try smaller values of the base b. First use the calculator to plot the graph of $y_1(x) = (1/2)^x$ (see **Figure 7**(a)).



(a) Graph of $y_1(x) = (1/2)^x$

(b) Comparing functions $y_1(x) = (1/2)^x$, $y_2(x) = (1/3)^x$, and $y_3(x) = (1/4)^x$

Figure 7.

This graph is much different. It rises rapidly to the left, and almost dies out on the right. Compare this with $y_2(x) = (1/3)^x$ and $y_3(x) = (1/4)^x$ (see **Figure 7**(b)). As the base gets smaller, the graph rises faster on the left, and dies out faster on the right.

Using reflection properties, it's easy to understand the appearance of these last three graphs. Note that

$$\left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}, \tag{18}$$

so it follows that the graph of $\left(\frac{1}{2}\right)^x$ is just a reflection in the y-axis of the graph of 2^x (see **Figure 8**).

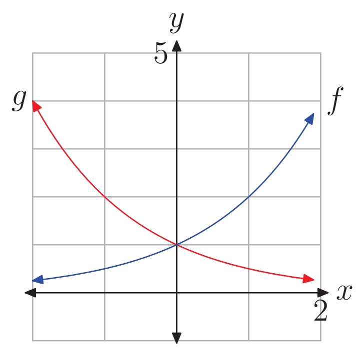


Figure 8. Comparing functions $f(x) = 2^x$ and $g(x) = (1/2)^x = 2^{-x}$

Thus, we seem to have two different types of graphs, and therefore two types of exponential functions: one type is increasing, and the other decreasing. Our experiments above, along with a little more experimentation, should convince you that b^x is increasing for $b > 1$, and decreasing for $0 < b < 1$. The first type of functions are called *exponential growth* functions, and the second type are *exponential decay* functions.

Properties of Exponential Growth Functions: $f(x) = b^x$ with $b > 1$

- The domain is the set of all real numbers.
- Moving from left to right, the graph rises, which means that the function increases as x increases. The function increases rapidly for positive x.
- The graph lies above the x-axis, so the values of the function are always positive. Therefore, the range is $(0, \infty)$.
- The graph has a horizontal asymptote $y = 0$ (the x-axis) on the left side. This means that the function almost "dies out" (the values get closer and closer to 0) as x approaches $-\infty$.

The second property above deserves some additional explanation. Looking at **Figure 6**(b), it appears that y_2 and y_3 increase rapidly as x increases, but y_1 appears to increase slowly. However, this is due to the fact that the graph of $y_1(x) = 2^x$

is only shown on the interval $[-2, 2]$. In **Figure 5**, the same function is graphed on the interval $[-4, 4]$, and it certainly appears to increase rapidly in that graph. The point here is that exponential growth functions *eventually* increase rapidly as x increases. If you graph the function on a large enough interval, the function will eventually become very steep on the right side of the graph. This is an important property of the exponential growth functions, and will be explored further in the exercises.

Properties of Exponential Decay Functions: $f(x) = b^x$ with $0 < b < 1$

- The domain is the set of all real numbers.
- Moving from left to right, the graph falls, which means that the function decreases as x increases. The function decreases rapidly for negative x.
- The graph lies above the x-axis, so the values of the function are always positive. Therefore, the range is $(0, \infty)$.
- The graph has a horizontal asymptote $y = 0$ (the x-axis) on the right side. This means that the function almost "dies out" (the values get closer and closer to 0) as x approaches ∞.

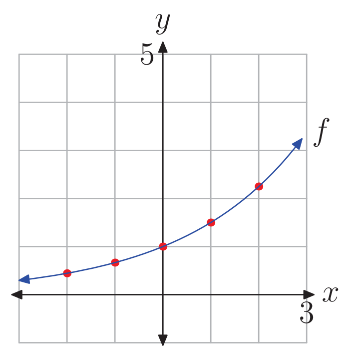
Why do we refrain from using the base $b = 1$? After all, 1^x is certainly defined: it has the value 1 for all x. But that means that $f(x) = 1^x$ is just a constant linear function – its graph is a horizontal line. Therefore, this function doesn't share the same properties as the other exponential functions, and we've already classified it as a linear function. Thus, 1^x is not considered to be an exponential function.

I **Example 19.** *Plot the graph of the function $f(x) = (1.5)^x$. Identify the range of the function and the horizontal asymptote.*

Since the base 1.5 is larger than 1, this is an exponential growth function. Therefore, its graph will have a shape similar to the graphs in **Figure 6**. The graph rises, there will be a horizontal asymptote $y = 0$ on the left side, and the range of the function is $(0, \infty)$. The graph can then be plotted by hand by using this knowledge along with approximate values at $x = -2, -1, 0, 1, 2$. See **Figure 9**.

x	$f(x) = (1.5)^x$
-2	0.44
-1	0.67
0	1
1	1.5
2	2.25

(a)



(b)

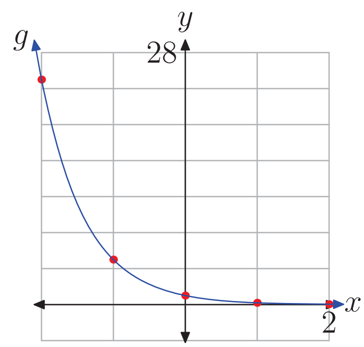
Figure 9. Graph of $f(x) = (1.5)^x$

I **Example 20.** *Plot the graph of the function $g(x) = (0.2)^x$. Identify the range of the function and the horizontal asymptote.*

Since the base 0.2 is smaller than 1, this is an exponential decay function. Therefore, its graph will have a shape similar to the graphs in **Figure 7**. The graph falls, there will be a horizontal asymptote $y = 0$ on the right side, and the range of the function is $(0, \infty)$. The graph can then be plotted by hand by using this knowledge along with approximate values at $x = -2, -1, 0, 1, 2$. See **Figure 10**.

x	$g(x) = (0.2)^x$
-2	25
-1	5
0	1
1	0.2
2	0.04

(a)



(b)

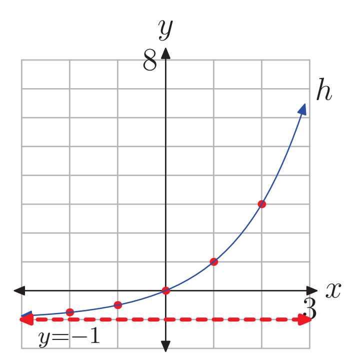
Figure 10. Graph of $g(x) = (0.2)^x$

I **Example 21.** *Plot the graph of the function $h(x) = 2^x - 1$. Identify the range of the function and the horizontal asymptote.*

The graph of h can be obtained from the graph of $f(x) = 2^x$ (see **Figure 5**) by a vertical shift down 1 unit. Therefore, the horizontal asymptote $y = 0$ of the graph of f will also be shifted down 1 unit, so the graph of h has a horizontal asymptote $y = -1$. Similarly, the range of f will be shifted down to $(-1, \infty) = \text{Range}(h)$. The graph can then be plotted by hand by using this knowledge along with approximate values at $x = -2, -1, 0, 1, 2$. See **Figure 11**.

x	$h(x) = 2^x - 1$
-2	-0.75
-1	-0.5
0	0
1	1
2	3

(a)



(b)

Figure 11. Graph of $h(x) = 2^x - 1$

In later sections of this chapter, we will also see more general exponential functions of the form $f(x) = Ab^x$ (in fact, the Pleasantville and Ghosttown functions at the beginning of this section are of this form). If A is positive, then the graphs of these functions can be obtained from the basic exponential graphs by vertical scaling, so the graphs will have the same general shape as either the exponential growth curves (if $b > 1$) or the exponential decay curves (if $0 < b < 1$) we plotted earlier.

4.3 *Exercises*

1. The current population of Fortuna is 10,000 hearty souls. It is known that the population is growing at a rate of 4% per year. Assuming this rate remains constant, perform each of the following tasks.

a. Set up an equation that models the population $P(t)$ as a function of time t.
b. Use the model in the previous part to predict the population 40 years from now.
c. Use your calculator to sketch the graph of the population over the next 40 years.

2. The population of the town of Imagination currently numbers 12,000 people. It is known that the population is growing at a rate of 6% per year. Assuming this rate remains constant, perform each of the following tasks.

a. Set up an equation that models the population $P(t)$ as a function of time t.
b. Use the model in the previous part to predict the population 30 years from now.
c. Use your calculator to sketch the graph of the population over the next 30 years.

3. The population of the town of Despairia currently numbers 15,000 individuals. It is known that the population is decaying at a rate of 5% per year. Assuming this rate remains constant, perform each of the following tasks.

a. Set up an equation that models the population $P(t)$ as a function of time t.
b. Use the model in the previous part to predict the population 50 years from now.
c. Use your calculator to sketch the graph of the population over the next 50 years.

4. The population of the town of Hopeless currently numbers 25,000 individuals. It is known that the population is decaying at a rate of 6% per year. Assuming this rate remains constant, perform each of the following tasks.

a. Set up an equation that models the population $P(t)$ as a function of time t.
b. Use the model in the previous part to predict the population 40 years from now.
c. Use your calculator to sketch the graph of the population over the next 40 years.

In **Exercises 5-12**, perform each of the following tasks for the given function.

a. Find the y-intercept of the graph of the function. Also, use your calculator to find two points on the graph to the right of the y-axis, and two points to the left.
b. Using your five points from (a) as a guide, set up a coordinate system on graph paper. Choose and label appropriate scales for each axis. Plot the five points, and any additional points you feel are necessary to dis-

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cern the shape of the graph.

c. Draw the horizontal asymptote with a dashed line, and label it with its equation.

d. Sketch the graph of the function.

e. Use interval notation to describe both the domain and range of the function.

5. $f(x) = (2.5)^x$

6. $f(x) = (0.1)^x$

7. $f(x) = (0.75)^x$

8. $f(x) = (1.1)^x$

9. $f(x) = 3^x + 1$

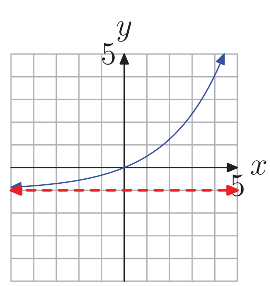
10. $f(x) = 4^x - 5$

11. $f(x) = 2^x - 3$

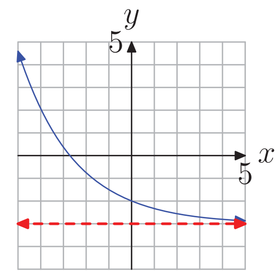
12. $f(x) = 5^x + 2$

In **Exercises 13-20**, the graph of an exponential function of the form $f(x) = b^x + c$ is shown. The dashed red line is a horizontal asymptote. Determine the range of the function. Express your answer in interval notation.

13.



14.



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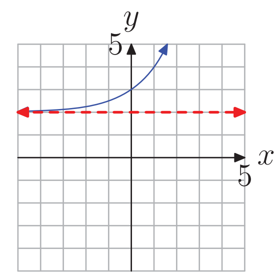

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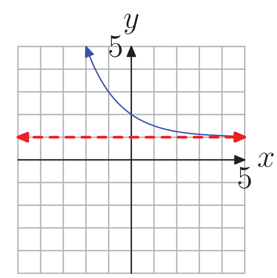

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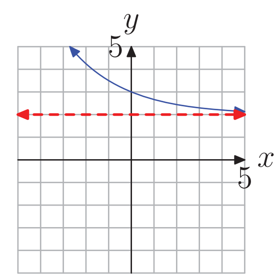
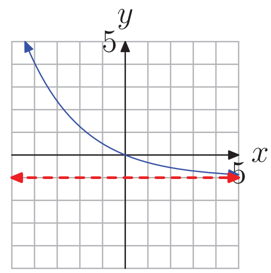
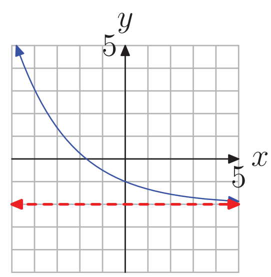


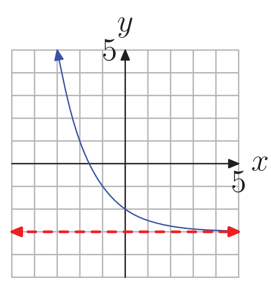
18.



19.



20.



In **Exercises 21-32**, compute $f(p)$ at the given value p.

21. $f(x) = (1/3)^x$; $p = -4$

22. $f(x) = (3/4)^x$; $p = 1$

23. $f(x) = 5^x$; $p = 5$

24. $f(x) = (1/3)^x$; $p = 4$

25. $f(x) = 4^x$; $p = -4$

26. $f(x) = 5^x$; $p = -3$

27. $f(x) = (5/2)^x$; $p = -3$

28. $f(x) = 9^x$; $p = 3$

29. $f(x) = 5^x$; $p = -4$

30. $f(x) = 9^x$; $p = 0$

31. $f(x) = (6/5)^x$; $p = -4$

32. $f(x) = (3/5)^x$; $p = 0$

In **Exercises 33-40**, use your calculator to evaluate the function at the given value p. Round your answer to the nearest hundredth.

33. $f(x) = 10^x$; $p = -0.7$.

34. $f(x) = 10^x$; $p = -1.60$.

35. $f(x) = (2/5)^x$; $p = 3.67$.

36. $f(x) = 2^x$; $p = -3/4$.

37. $f(x) = 10^x$; $p = 2.07$.

38. $f(x) = 7^x$; $p = 4/3$.

39. $f(x) = 10^x$; $p = -1/5$.

40. $f(x) = (4/3)^x$; $p = 1.15$.

41. This exercise explores the property that exponential growth functions *eventually* increase rapidly as x increases. Let $f(x) = 1.05^x$. Use your graphing calculator to graph f on the intervals

(a) $[0, 10]$ and (b) $[0, 100]$.

For (a), use Ymin = 0 and Ymax = 10. For (b), use Ymin = 0 and Ymax = 100. Make accurate copies of the images in your viewing window on your homework paper. What do you observe when you compare the two graphs?

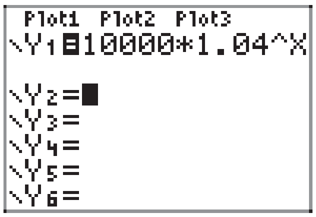
4.3 Answers

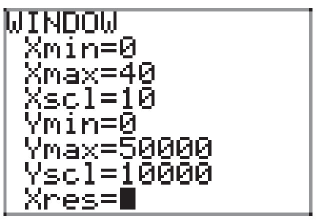
1.

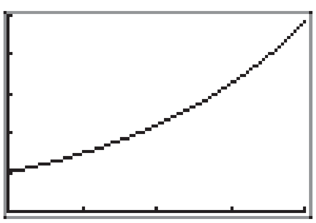
a) $P(t) = 10\,000(1.04)^t$

b) $P(40) \approx 48\,101$

c)





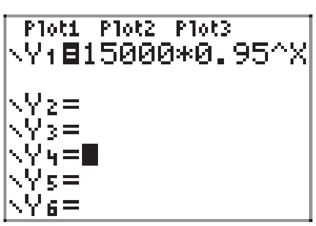


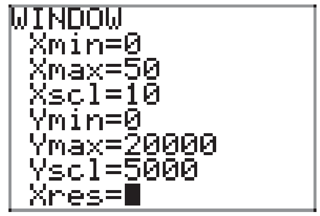
3.

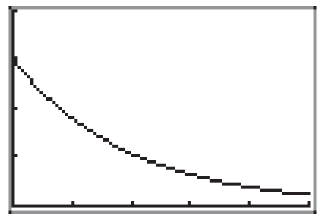
a) $P(t) = 15\,000(0.95)^t$

b) $P(50) \approx 1\,154$

c)







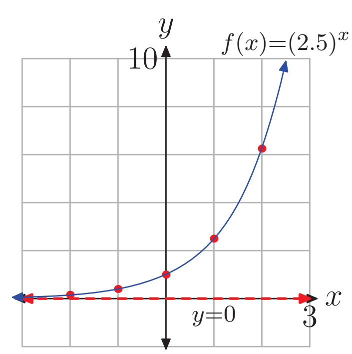
5.

a) The y-intercept is $(0, 1)$. Evaluate the function at $x = 1, 2, -1, -2$ to obtain the points $(1, 2.5)$, $(2, 6.25)$, $(-1, 0.4)$, $(-2, 0.16)$ (other answers are possible).

b) See the graph in part (d).

c) The horizontal asymptote is $y = 0$. See the graph in part (d).

d)



e) Domain $= (-\infty, \infty)$, Range $= (0, \infty)$

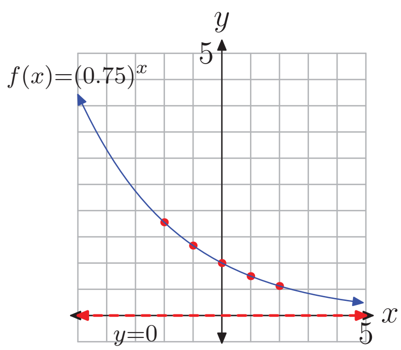
7.

a) The y-intercept is $(0, 1)$. Evaluate the function at $x = 1, 2, -1, -2$ to obtain the points $(1, 0.75)$, $(2, 0.56)$, $(-1, 1.34)$, $(-2, 1.78)$ (other answers are possible).

b) See the graph in part (d).

c) The horizontal asymptote is $y = 0$. See the graph in part (d).

d)



e) Domain $= (-\infty, \infty)$, Range $= (0, \infty)$

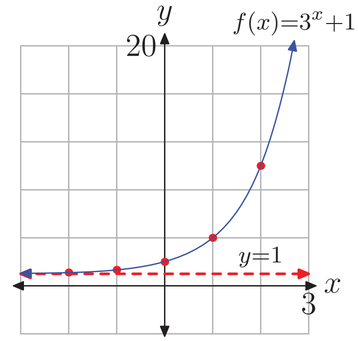
9.

a) The y-intercept is $(0, 2)$. Evaluate the function at $x = 1, 2, -1, -2$ to obtain the points $(1, 4)$, $(2, 10)$, $(-1, 1.34)$, $(-2, 1.11)$ (other answers are possible).

b) See the graph in part (d).

c) The horizontal asymptote is $y = 1$. See the graph in part (d).

d)



e) Domain $= (-\infty, \infty)$, Range $= (1, \infty)$

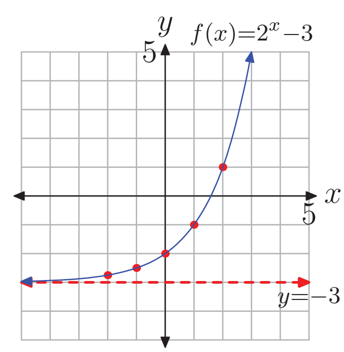
11.

a) The y-intercept is $(0, -2)$. Evaluate the function at $x = 1, 2, -1, -2$ to obtain the points $(1, -1)$, $(2, 1)$, $(-1, -2.5)$, $(-2, -2.75)$ (other answers are possible).

b) See the graph in part (d).

c) The horizontal asymptote is $y = -3$. See the graph in part (d).

d)



e) Domain $= (-\infty, \infty)$, Range $= (-3, \infty)$

13. $(-1, \infty)$

15. $(2, \infty)$

17. $(2, \infty)$

19. $(-2, \infty)$

21. 81

23. 3125

25. $\dfrac{1}{256}$

27. $\dfrac{8}{125}$

29. $\dfrac{1}{625}$

31. $\dfrac{625}{1296}$

33. 0.20

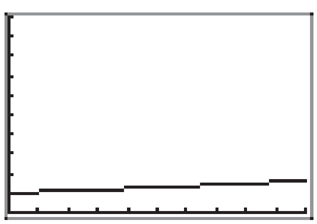
35. 0.03

37. 117.49

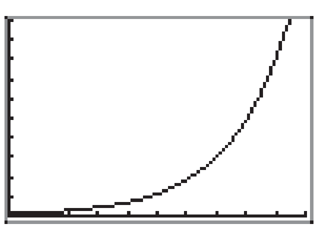
39. 0.63

41.

a) The graph on the interval $[0, 10]$ increases very slowly. In fact, the graph looks almost linear.



b) The graph on the interval $[0, 100]$ increases slowly at first, but then increases very rapidly on the second half of the interval.



4.4 Applications of Exponential Functions

In the preceding section, we examined a population growth problem in which the population grew at a fixed percentage each year. In that case, we found that the population can be described by an exponential function. A similar analysis will show that any process in which a quantity grows by a fixed percentage each year (or each day, hour, etc.) can be modeled by an exponential function. Compound interest is a good example of such a process.

Discrete Compound Interest

If you put money in a savings account, then the bank will pay you interest (a percentage of your account balance) at the end of each time period, typically one month or one day. For example, if the time period is one month, this process is called *monthly compounding*. The term compounding refers to the fact that interest is added to your account each month and then in subsequent months you earn interest on the interest. If the time period is one day, it's called *daily compounding*.

Let's look at monthly compounding in more detail. Suppose that you deposit \$100 in your account, and the bank pays interest at an annual rate of 5%. Let the function $P(t)$ represent the amount of money that you have in your account at time t, where we measure t in years. We will start time at $t = 0$ when the initial amount, called the *principal*, is \$100. In other words, $P(0) = 100$.

In the discussion that follows, we will compute the account balance at the end of each month. Since one month is 1/12 of a year, $P(1/12)$ represents the balance at the end of the first month, $P(2/12)$ represents the balance at the end of the second month, etc.

At the end of the first month, interest is added to the account balance. Since the annual interest rate 5%, the monthly interest rate is 5%/12, or .05/12 in decimal form. Although we could approximate .05/12 by a decimal, it will be more useful, as well as more accurate, to leave it in this form. Therefore, at the end of the first month, the interest earned will be 100(.05/12), so the total amount will be

$$P(1/12) = 100 + 100\left(\frac{.05}{12}\right) = 100\left(1 + \frac{.05}{12}\right). \tag{1}$$

Now at the end of the second month, you will have the amount that you started that month with, namely $P(1/12)$, plus another month's worth of interest on that amount. Therefore, the total amount will be

$$P(2/12) = P(1/12) + P(1/12)\left(\frac{.05}{12}\right) = P(1/12)\left(1 + \frac{.05}{12}\right). \tag{2}$$

If we replace $P(1/12)$ in **equation (2)** with the result found in **equation (1)**, then

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$$P(2/12) = 100\left(1+\frac{.05}{12}\right)\left(1+\frac{.05}{12}\right) = 100\left(1+\frac{.05}{12}\right)^2. \tag{3}$$

Let's iterate one more month. At the end of the third month, you will have the amount that you started that month with, namely $P(2/12)$, plus another month's worth of interest on that amount. Therefore, the total amount will be

$$P(3/12) = P(2/12) + P(2/12)\left(\frac{.05}{12}\right) = P(2/12)\left(1+\frac{.05}{12}\right). \tag{4}$$

However, if we replace $P(2/12)$ in **equation (4)** with the result found in **equation (3)**, then

$$P(3/12) = 100\left(1+\frac{.05}{12}\right)^2\left(1+\frac{.05}{12}\right) = 100\left(1+\frac{.05}{12}\right)^3. \tag{5}$$

The pattern should now be clear. The amount of money you will have in the account at the end of m months is given by the function

$$P(m/12) = 100\left(1+\frac{.05}{12}\right)^m.$$

We can rewrite this formula in terms of years t by replacing $m/12$ by t. Then $m = 12t$, so the formula becomes

$$P(t) = 100\left(1+\frac{.05}{12}\right)^{12t}. \tag{6}$$

What would be different if you had started with a principal of 200? By tracing over our previous steps, it should be easy to see that the new formula would be

$$P(t) = 200\left(1+\frac{.05}{12}\right)^{12t}.$$

Similarly, if the interest rate had been 4% per year instead of 5%, then we would have ended up with the formula

$$P(t) = 100\left(1+\frac{.04}{12}\right)^{12t}.$$

Thus, if we let P_0 represent the principal, and r represent the annual interest rate (in decimal form), then we can generalize the formula to

$$P(t) = P_0\left(1+\frac{r}{12}\right)^{12t}. \tag{7}$$

I **Example 8.** *If the principal is \$100, the annual interest rate is 5%, and interest is compounded monthly, how much money will you have after ten years?*

In formula (**7**), let $P_0 = 100$, $r = .05$, and $t = 10$:

$$P(10) = 100\left(1+\frac{.05}{12}\right)^{12\cdot 10}$$

We can use our graphing calculator to approximate this solution, as shown in **Figure 1**.

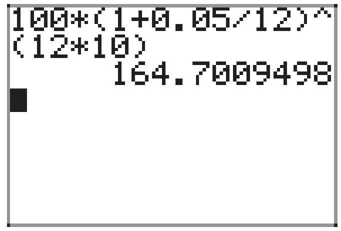


Figure 1. Computing the amount after compounding monthly for 10 years.

Thus, you would have \$164.70 after ten years.

I **Example 9.** *If the principal is \$10 000, the annual interest rate is 5%, and interest is compounded monthly, how much money will you have after forty years?*

In formula (**7**), let $P_0 = 10\,000$, $r = .05$, and $t = 40$:

$$P(40) = 10\,000\left(1 + \frac{.05}{12}\right)^{12\cdot 40} \approx 73\,584.17$$

Thus, you would have \$73,584.17 after forty years.

These examples illustrate the "miracle of compound interest." In the last example, your account is more than seven times as large as the original, and your total "profit" (the amount of interest you've received) is \$63 584.17. Compare this to the amount you would have received if you had withdrawn the interest each month (i.e., no compounding). In that case, your "profit" would only be \$20 000:

$$\text{years}\cdot\frac{\text{months}}{\text{year}}\cdot\frac{\text{interest}}{\text{month}} = 40\cdot 12\cdot\left[(10\,000)\left(\frac{.05}{12}\right)\right] = 20\,000$$

The large difference can be attributed to the shape of the graph of the function $P(t)$. Recall from the preceding section that this is an exponential growth function, so as t gets large, the graph will eventually rise steeply. Thus, if you can leave your money in the bank long enough, it will eventually grow dramatically.

What about daily compounding? Let's again analyze the situation in which the principal is \$100 and the annual interest rate is 5%. In this case, the time period over which interest is paid is one day, or 1/365 of a year, and the daily interest rate is 5%/365, or .05/365 in decimal form. Since we are measuring time in years, $P(1/365)$ represents the balance at the end of the first day, $P(2/365)$ represents the balance at the end of the second day, etc. We'll follow the same steps as in the earlier analysis for monthly compounding.

At the end of the first day, you will have

$$P(1/365) = 100 + 100\left(\frac{.05}{365}\right) = 100\left(1 + \frac{.05}{365}\right). \tag{10}$$

At the end of the second day, you will have the amount that you started that day with, namely $P(1/365)$, plus another day's worth of interest on that amount. Therefore, the total amount will be

$$P(2/365) = P(1/365) + P(1/365)\left(\frac{.05}{365}\right) = P(1/365)\left(1 + \frac{.05}{365}\right). \tag{11}$$

If we replace $P(1/365)$ in **equation (11)** with the result found in **equation (10)**, then

$$P(2/365) = 100\left(1 + \frac{.05}{365}\right)\left(1 + \frac{.05}{365}\right) = 100\left(1 + \frac{.05}{365}\right)^2. \tag{12}$$

At the end of the third day, you will have the amount that you started that day with, namely $P(2/365)$, plus another day's worth of interest on that amount. Therefore, the total amount will be

$$P(3/365) = P(2/365) + P(2/365)\left(\frac{.05}{365}\right) = P(2/365)\left(1 + \frac{.05}{365}\right). \tag{13}$$

Again, replacing $P(2/365)$ in **equation (13)** with the result found in **equation (12)** yields

$$P(3/365) = 100\left(1 + \frac{.05}{365}\right)^2\left(1 + \frac{.05}{365}\right) = 100\left(1 + \frac{.05}{365}\right)^3. \tag{14}$$

Continuing this pattern shows that the amount of money you will have in the account at the end of d days is given by the function

$$P(d/365) = 100\left(1 + \frac{.05}{365}\right)^d.$$

We can rewrite this formula in terms of years t by replacing $d/365$ by t. Then $d = 365t$, so the formula becomes

$$P(t) = 100\left(1 + \frac{.05}{365}\right)^{365t}. \tag{15}$$

More generally, if you had started with a principal of P_0 and an annual interest rate of r (in decimal form), then the formula would be

$$P(t) = P_0\left(1 + \frac{r}{365}\right)^{365t}. \tag{16}$$

Comparing formulas (**7**) and (**16**) for monthly and daily compounding, it should be apparent that the only difference is that the number 12 is used in the monthly compounding formula and the number 365 is used in the daily compounding formula. Looking at the respective analyses shows that this number arises from the portion of the year that interest is paid (1/12 in the case of monthly compounding, and 1/365 in the case of daily compounding). Thus, in each case, this number (12 or 365) also equals the number of times that interest is compounded per year. It follows that if interest is compounded quarterly (every three months, or 4 times per year), the formula would be

$$P(t) = P_0\left(1 + \frac{r}{4}\right)^{4t}.$$

Similarly, if interest is compounded hourly (8760 times per year), the formula would be

$$P(t) = P_0 \left(1 + \frac{r}{8760}\right)^{8760t}.$$

Summarizing, we have one final generalization:

Discrete Compound Interest

If P_0 is the principal, r is the annual interest rate, and n is the number of times that interest is compounded per year, then the balance at time t years is

$$P(t) = P_0 \left(1 + \frac{r}{n}\right)^{nt}. \tag{17}$$

I **Example 18.** *If the principal is \$100, the annual interest rate is 5%, and interest is compounded daily, what will be the balance after ten years?*

In formula (**17**), let $P_0 = 100$, $r = .05$, $n = 365$, and $t = 10$:

$$P(10) = 100 \left(1 + \frac{.05}{365}\right)^{365 \cdot 10} \approx 164.87$$

Thus, you would have \$164.87 after ten years.

I **Example 19.** *If the principal is \$10 000, the annual interest rate is 5%, and interest is compounded daily, what will be the balance after forty years?*

In formula (**17**), let $P_0 = 10\,000$, $r = .05$, $n = 365$, and $t = 40$:

$$P(40) = 10\,000 \left(1 + \frac{.05}{365}\right)^{365 \cdot 40} \approx 73\,880.44$$

Thus, you would have \$73 880.44 after forty years.

As you can see from comparing Examples **8** and **18**, and Examples **9** and **19**, the difference between monthly and daily compounding is generally small. However, the difference can be substantial for large principals and/or large time periods.

I **Example 20.** *If the principal is \$500, the annual interest rate is 8%, and interest is compounded quarterly, what will be the balance after 42 months?*

42 months is 3.5 years, so let $P_0 = 500$, $r = .08$, $n = 4$, and $t = 3.5$ in formula (**17**):

$$P(5) = 500 \left(1 + \frac{.08}{4}\right)^{4 \cdot 3.5} \approx 659.74$$

Thus, you would have \$659.74 after 42 months.

Continuous Compound Interest and the Number e

Using formula (**17**), it is a simple matter to calculate the total amount for any type of compounding. Although most banks compound interest either daily or monthly, it could be done every hour, or every minute, or every second, etc. What happens to the total amount as the time period shortens? Equivalently, what happens as n increases in formula (**17**)? **Table 1** shows the amount after one year with a principal of $P_0 = 100$, $r = .05$, and various values of n:

compounding	n	$P(1)$
monthly	12	105.11619
daily	365	105.12675
hourly	8760	105.12709
every minute	525600	105.12711
every second	31536000	105.12711

Table 1. Comparison of discrete compounding with $P_0 = 100$, $r = .05$, and $t = 1$ year.

Even if we carry out our computations to eight digits, it appears that the amounts in the right hand column of **Table 1** are stabilizing. In fact, using calculus, it can be shown that these amounts do indeed get closer and closer to a particular number, and we can calculate that number.

Starting with formula (**17**), we will let n approach ∞. In other words, we will let n get larger and larger without bound, as we started to do in **Table 1**. The first step is to use the Laws of Exponents to write

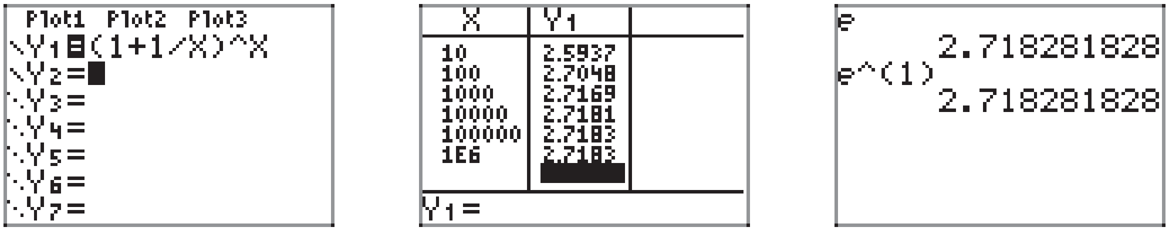
$$P_0\left(1+\frac{r}{n}\right)^{nt} = P_0\left[\left(1+\frac{r}{n}\right)^{\frac{n}{r}}\right]^{rt}.$$

In the next step, replace n/r by m. Since $n/r = m$, it follows that $r/n = 1/m$, and we have

$$P_0\left[\left(1+\frac{r}{n}\right)^{\frac{n}{r}}\right]^{rt} = P_0\left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt}.$$

Now let n approach ∞. Since $m = n/r$ and r is fixed, it follows that m also approaches ∞. We can use the `TABLE` feature of the graphing calculator to investigate the convergence of the expression in brackets as m approaches infinity.

- Load $(1+1/m)^m$ into the `Y=` menu of the graphing calculator, as shown in **Figure 2**(a). Of course, you must use x instead of m and enter `(1+1/X)^X`.
- Use `TBLSET` and set `Indepnt` to `Ask`, select `TABLE`, then enter the numbers 10, 100, 1 000, 10 000, 100 000, and 1 000 000 to produce the result shown in **Figure 2**(b). Note that `(1+1/X)^X` appears to converge to the number 2.7183. If you move the cursor over the last result in the `Y1` column, you can see more precision, 2.71828046932.



(a) (b) (c)

Figure 2. Illustration of the convergence of $(1+1/m)^m$ to e as m increases to infinity

Note that the numbers in the second column in **Figure 2**(b) appear to stabilize. Indeed, it can be shown by using calculus that the expression in brackets above gets closer and closer to a single number, which is called e. To represent this convergence, we write

$$\left(1+\frac{1}{m}\right)^m \to e. \tag{21}$$

e is an irrational number, approximately 2.7183, as shown by the computations in **Figure 2**(b). It follows that

$$P_0\left[\left(1+\frac{1}{m}\right)^m\right]^{rt} \to P_0e^{rt}.$$

Because we took the discrete compound interest formula (**17**) and let the number of times compounded per year (n) approach ∞, this process is known as *continuous compounding*.

Continuous Compound Interest

If P_0 is the principal, r is the annual interest rate, and interest is compounded continuously, then the balance at time t years is

$$P(t) = P_0e^{rt}. \tag{22}$$

Before working the next examples, find the buttons on your calculator for the number e and for the exponential function e^x. Typing either `e` or `e^(1)` (using the e^x button) will yield an approximation to the number e, as shown in **Figure 2**(c). Compare this approximation with the one you obtained earlier in **Figure 2**(b).

I **Example 23.** *If the principal is \$100, the annual interest rate is 5%, and interest is compounded continuously, what will be the balance after ten years?*

In formula (**22**), let $P_0 = 100$, $r = 0.05$, and $t = 10$:

$$P(10) = 100e^{(0.05)(10)}$$

Use your calculator to approximate this result, as shown in **Figure 3**.

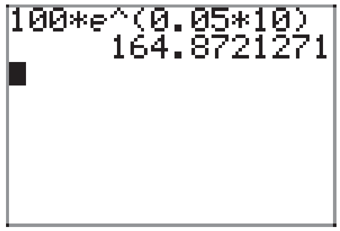


Figure 3. Computing the amount after compounding continuously for 10 years.

Thus, you would have \$164.87 after ten years.

I **Example 24.** *If the principal is \$10,000, the annual interest rate is 5%, and interest is compounded continuously, what will be the balance after forty years?*

In formula (**22**), let $P_0 = 10\,000$, $r = 0.05$, and $t = 40$:

$$P(10) = 10\,000e^{(0.05)(40)} \approx 73\,890.56$$

Thus, you would have \$73 890.56 after forty years.

Notice that the continuous compounding formula (**22**) is much simpler than the discrete compounding formula (**17**). Unless the principal is very large or the time period is very long, the preceding examples show that continuous compounding is also a close approximation to daily compounding. In **Example 23**, the amount \$164.87 is the same (rounded to the nearest cent) as the amount for daily compounding found in **Example 18**. With a larger principal and longer time period, the amount \$73 890.56 in **Example 24** using continuous compounding is still only about \$10 more than the amount \$73 880.44 for daily compounding found in **Example 19**.

Remarks 25.

1. The number e may strike you as a mere curiosity. If so, that would be a big misconception. The number e is actually one of the most important numbers in mathematics (it's probably the second most famous number, following π), and it arises naturally as the limit described in (**21**) above. Using notation from calculus, we write

$$\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = e \approx 2.71828. \tag{26}$$

   Although in our discussion above this limit arose in a man-made process, compound interest, it shows up in a similar manner in studies of many natural phenomena. We'll look at some of these applications later in this chapter.
2. Likewise, the exponential function e^x is one of the most important functions used in mathematics, statistics, and many fields of science. For a variety of reasons, the base e turns out to be the most natural base to use for an exponential function. Consequently, the function $f(x) = e^x$ is known as the *natural exponential function*.

Future Value and Present Value

In this section we have derived two formulas, one for discrete compound interest, and the other for continuous compound interest. However, in the examples presented so far, we've only used these formulas to calculate *future value*: given a principal P_0 and interest rate r, how much will you have in your account in t years?

Another type of question we can solve is known as a *present value* problem: how much money would you have to invest at interest r in order to have Q dollars in t years? Here are a couple of examples:

I **Example 27.** *How much money would you have to invest at 4% interest compounded daily in order to have \$8000 dollars in* 6 *years?*

In this case, the principal P_0 is unknown, and we substitute $r = 0.04$, $n = 365$, and $t = 6$, into the discrete compounding formula (**17**). Since $P(6) = 8000$, we have the equation

$$8000 = P(6) = P_0 \left(1 + \frac{0.04}{365}\right)^{(365)(6)}.$$

This equation can be solved by division:

$$\frac{8000}{\left(1 + \frac{0.04}{365}\right)^{(365)(6)}} = P_0$$

Figure 4 shows a calculator approximation for this result.

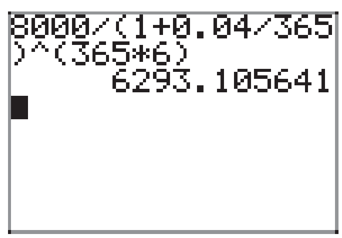


Figure 4. The present value of \$8000, compounded daily for six years.

Thus, the present value is approximately $P_0 \approx \$6293.11$. If this amount is invested now at 4% compounded daily, then its future value in 6 years will be \$8000.

I **Example 28.** *How much money would you have to invest at 7% interest compounded continuously in order to have \$5000 dollars in* 4 *years?*

As in the last example, the principal P_0 is unknown, and this time $r = 0.07$ and $t = 4$ in the continuous compounding formula (**22**). Then $P(4) = 5000$ yields the equation

$$5000 = P(4) = P_0 e^{(0.07)(4)}.$$

As in the last example, this equation can also be solved by division:

$$\frac{5000}{e^{(0.07)(4)}} = P_0$$

A calculator approximation for this result is shown in **Figure 5**.

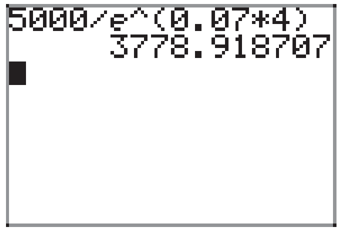


Figure 5. The present value of \$5000, compounded continuously for four years.

Thus, the present value is approximately $P_0 \approx \$3778.92$. If this amount is invested now at 7% compounded continuously, then its future value in 4 years will be \$5000.

Additional Questions

In terms of practical applications, there are also other types of questions that would be interesting to consider. Here are two examples:

1. If you deposit \$1000 in an account paying 6% compounded continuously, how long will it take for you to have \$1500 in your account?
2. If you deposit \$1000 in an account paying 5% compounded monthly, how long will it take for your money to double?

Let's look at the first question (the second is similar). In this case, $P_0 = 1000$ and $r = 0.06$. Inserting these values into the continuous compounding formula (**22**), we obtain

$$P(t) = 1000e^{0.06t}.$$

Now we want the future value $P(t)$ of the account at some time t to equal \$1500. Therefore, we must solve the equation

$$1500 = 1000e^{0.06t}.$$

However, now we have a problem, because the variable t is located in the exponent of the expression on the right side of the equation. Although we could approximate a solution graphically, we currently have no algebraic method for solving an equation such as this, where the variable is in the exponent (these types of equations are called *exponential equations*). Over the course of the next few sections, we will define another type of function, the logarithm function, which will in turn provide us with a method for solving exponential equations. Then we will return to these questions, and also discuss additional applications.

4.4 *Exercises*

1. Suppose that you invest \$15,000 at 7% interest compounded monthly. How much money will be in your account in 4 years? Round your answer to the nearest cent.

2. Suppose that you invest \$14,000 at 3% interest compounded monthly. How much money will be in your account in 7 years? Round your answer to the nearest cent.

3. Suppose that you invest \$14,000 at 4% interest compounded daily. How much money will be in your account in 6 years? Round your answer to the nearest cent.

4. Suppose that you invest \$15,000 at 8% interest compounded monthly. How much money will be in your account in 8 years? Round your answer to the nearest cent.

5. Suppose that you invest \$4,000 at 3% interest compounded monthly. How much money will be in your account in 7 years? Round your answer to the nearest cent.

6. Suppose that you invest \$3,000 at 5% interest compounded monthly. How much money will be in your account in 4 years? Round your answer to the nearest cent.

7. Suppose that you invest \$1,000 at 3% interest compounded monthly. How much money will be in your account in 4 years? Round your answer to the nearest cent.

8. Suppose that you invest \$19,000 at 2% interest compounded daily. How much money will be in your account in 9 years? Round your answer to the nearest cent.

9. Suppose that you can invest money at 4% interest compounded monthly. How much should you invest in order to have \$20,000 in 2 years? Round your answer to the nearest cent.

10. Suppose that you can invest money at 6% interest compounded daily. How much should you invest in order to have \$1,000 in 2 years? Round your answer to the nearest cent.

11. Suppose that you can invest money at 3% interest compounded daily. How much should you invest in order to have \$20,000 in 3 years? Round your answer to the nearest cent.

12. Suppose that you can invest money at 3% interest compounded monthly. How much should you invest in order to have \$10,000 in 7 years? Round your answer to the nearest cent.

13. Suppose that you can invest money at 9% interest compounded daily. How much should you invest in order to have \$4,000 in 9 years? Round your answer to the nearest cent.

14. Suppose that you can invest money at 8% interest compounded daily. How much should you invest in order to have \$18,000 in 6 years? Round your answer to the nearest cent.

15. Suppose that you can invest money at 8% interest compounded daily. How much should you invest in order to have $17,000 in 6 years? Round your answer to the nearest cent.

16. Suppose that you can invest money at 9% interest compounded daily. How much should you invest in order to have $5,000 in 7 years? Round your answer to the nearest cent.

In **Exercises 17-24**, evaluate the function at the given value p. Round your answer to the nearest hundredth.

17. $f(x) = e^x$; $p = 1.57$.

18. $f(x) = e^x$; $p = 2.61$.

19. $f(x) = e^x$; $p = 3.07$.

20. $f(x) = e^x$; $p = -4.33$.

21. $f(x) = e^x$; $p = 1.42$.

22. $f(x) = e^x$; $p = -0.8$.

23. $f(x) = e^x$; $p = 4.75$.

24. $f(x) = e^x$; $p = 3.60$.

25. Suppose that you invest $3,000 at 4% interest compounded continuously. How much money will be in your account in 9 years? Round your answer to the nearest cent.

26. Suppose that you invest $8,000 at 8% interest compounded continuously. How much money will be in your account in 7 years? Round your answer to the nearest cent.

27. Suppose that you invest $1,000 at 2% interest compounded continuously. How much money will be in your account in 3 years? Round your answer to the nearest cent.

28. Suppose that you invest $3,000 at 8% interest compounded continuously. How much money will be in your account in 4 years? Round your answer to the nearest cent.

29. Suppose that you invest $15,000 at 2% interest compounded continuously. How much money will be in your account in 4 years? Round your answer to the nearest cent.

30. Suppose that you invest $8,000 at 2% interest compounded continuously. How much money will be in your account in 6 years? Round your answer to the nearest cent.

31. Suppose that you invest $13,000 at 9% interest compounded continuously. How much money will be in your account in 8 years? Round your answer to the nearest cent.

32. Suppose that you invest $16,000 at 4% interest compounded continuously. How much money will be in your account in 6 years? Round your answer to the nearest cent.

33. Suppose that you can invest money at 6% interest compounded continuously. How much should you invest in order to have $17,000 in 9 years? Round your answer to the nearest cent.

34. Suppose that you can invest money at 8% interest compounded continuously. How much should you invest in order to have $5,000 in 6 years? Round your answer to the nearest cent.

35. Suppose that you can invest money at 8% interest compounded continuously. How much should you invest in order to have \$10,000 in 6 years? Round your answer to the nearest cent.

36. Suppose that you can invest money at 6% interest compounded continuously. How much should you invest in order to have \$17,000 in 13 years? Round your answer to the nearest cent.

37. Suppose that you can invest money at 2% interest compounded continuously. How much should you invest in order to have \$13,000 in 8 years? Round your answer to the nearest cent.

38. Suppose that you can invest money at 9% interest compounded continuously. How much should you invest in order to have \$10,000 in 15 years? Round your answer to the nearest cent.

39. Suppose that you can invest money at 7% interest compounded continuously. How much should you invest in order to have \$18,000 in 10 years? Round your answer to the nearest cent.

40. Suppose that you can invest money at 9% interest compounded continuously. How much should you invest in order to have \$14,000 in 12 years? Round your answer to the nearest cent.

4.4 Answers

1. $19830.81

3. $17797.25

5. $4933.42

7. $1127.33

9. $18464.78

11. $18278.69

13. $1779.61

15. $10519.87

17. 4.81

19. 21.54

21. 4.14

23. 115.58

25. $4299.99

27. $1061.84

29. $16249.31

31. $26707.63

33. $9906.72

35. $6187.83

37. $11077.87

39. $8938.54

4.5 Inverse Functions

As we saw in the last section, in order to solve application problems involving exponential functions, we will need to be able to solve exponential equations such as

$$1500 = 1000e^{0.06t} \qquad \text{or} \qquad 300 = 2^x.$$

However, we currently don't have any mathematical tools at our disposal to solve for a variable that appears as an exponent, as in these equations. In this section, we will develop the concept of an inverse function, which will in turn be used to define the tool that we need, the logarithm, in Section 8.5.

One-to-One Functions

Definition 1. *A given function f is said to be* one-to-one *if for each value y in the range of f, there is just one value x in the domain of f such that $y = f(x)$. In other words, f is one-to-one if each output y of f corresponds to precisely one input x.*

It's easiest to understand this definition by looking at mapping diagrams and graphs of some example functions.

I **Example 2.** *Consider the two functions h and k defined according to the mapping diagrams in* **Figure 1**. *In* **Figure 1***(a), there are two values in the domain that are both mapped onto 3 in the range. Hence, the function h is not one-to-one. On the other hand, in* **Figure 1***(b), for each output in the range of k, there is only one input in the domain that gets mapped onto it. Therefore, k is a one-to-one function.*

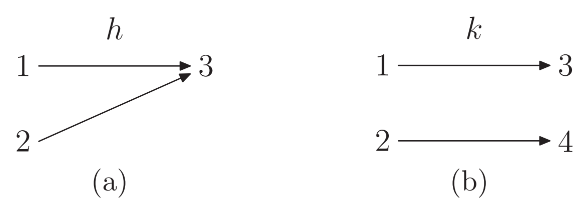
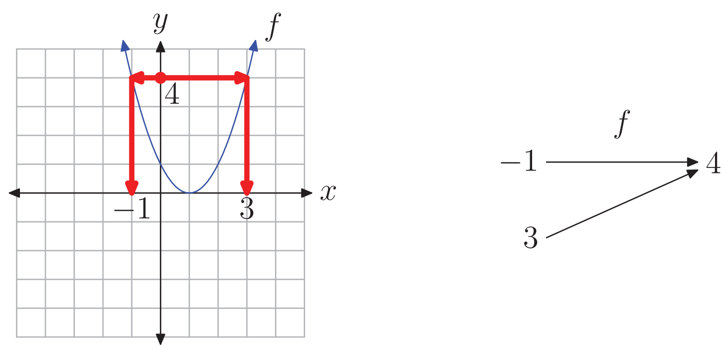


Figure 1. Mapping diagrams help to determine if a function is one-to-one.

I **Example 3.** *The graph of a function is shown in* **Figure 2***(a). For this function f, the y-value 4 is the output corresponding to two input values, $x = -1$ and $x = 3$ (see the corresponding mapping diagram in* **Figure 2***(b)). Therefore, f is* not *one-to-one.*

Graphically, this is apparent by drawing horizontal segments from the point $(0, 4)$ on the y-axis over to the corresponding points on the graph, and then drawing vertical segments to the x-axis. These segments meet the x-axis at -1 and 3.



(a) (b)

Figure 2. A function which is not one-to-one.

I **Example 4.** *In* **Figure 3***, each y-value in the range of f corresponds to just one input value x. Therefore, this function* is *one-to-one.*

Graphically, this can be seen by mentally drawing a horizontal segment from each point on the y-axis over to the corresponding point on the graph, and then drawing a vertical segment to the x-axis. Several examples are shown in **Figure 3**. It's apparent that this procedure will always result in just one corresponding point on the x-axis, because each y-value only corresponds to one point on the graph. In fact, it's easiest to just note that since each horizontal line only intersects the graph once, then there can be only one corresponding input to each output.

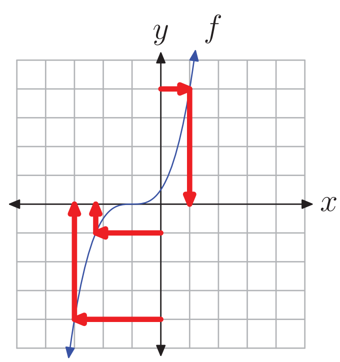


Figure 3. A one-to-one function

The graphical process described in the previous example, known as the *horizontal line test*, provides a simple visual means of determining whether a function is one-to-one.

Horizontal Line Test

If each horizontal line intersects the graph of f at most once, then f is one-to-one. On the other hand, if some horizontal line intersects the graph of f more than once, then f is *not* one-to-one.

Remark 5. It follows from the horizontal line test that if f is a strictly increasing function, then f is one-to-one. Likewise, every strictly decreasing function is also one-to-one.

Inverse Functions

If f is one-to-one, then we can define an associated function g, called the *inverse function* of f. We will give a formal definition below, but the basic idea is that the inverse function g simply sends the outputs of f back to their corresponding inputs. In other words, the mapping diagram for g is obtained by reversing the arrows in the mapping diagram for f.

I **Example 6.** *The function f in* **Figure 4***(a) maps 1 to 5 and 2 to -3. Therefore, the inverse function g in* **Figure 4***(b) maps the outputs of f back to their corresponding inputs: 5 to 1 and -3 to 2. Note that reversing the arrows on the mapping diagram for f yields the mapping diagram for g.*

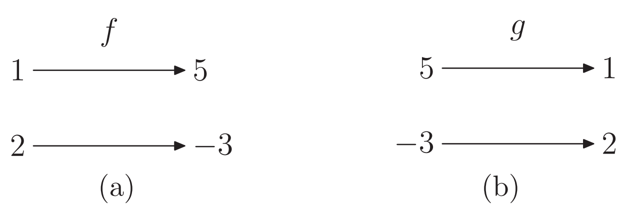


Figure 4. Reversing the arrows on the mapping diagram for f yields the mapping diagram for g.

Since the inverse function g sends the outputs of f back to their corresponding inputs, it follows that the inputs of g are the outputs of f, and vice versa. Thus, the functions g and f are related by simply interchanging their inputs and outputs.

The original function must be one-to-one in order to have an inverse. For example, consider the function h in **Example 2**. h is not one-to-one. If we reverse the arrows in the mapping diagram for h (see **Figure 1**(a)), then the resulting relation will not be a function, because 3 would map to both 1 and 2.

Before giving the formal definition of an inverse function, it's helpful to review the description of a function given in Section 2.1. While functions are often defined by means of a formula, remember that in general a function is just a *rule* that dictates how to associate a unique output value to each input value.

Definition 7. *Suppose that f is a given one-to-one function. The inverse function g is defined as follows: for each y in the range of f, define $g(y)$ to be the unique value x such that $y = f(x)$.*

To understand this definition, it's helpful to look at a diagram:

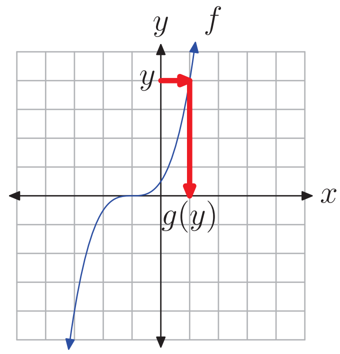


Figure 5.

The input for g is any y-value in the range of f. Thus, the input in the above diagram is a value on the y-axis. The output of g is the corresponding value on the x-axis which satisfies the condition $y = f(x)$. Note in particular that the x-value is unique because f is one-to-one.

The relationship between the original function f and its inverse function g can be described by:[10]

Property 8. If g is the inverse function of f, then

$$x = g(y) \quad \iff \quad y = f(x).$$

In fact, this is really the defining relationship for the inverse function. An easy way to understand this relationship (and the entire concept of an inverse function) is to realize that it states that *inputs and outputs are interchanged*. The inputs of g are the outputs of f, and vice versa. It follows that the Domain and Range of f and g are interchanged:

Property 9. If g is the inverse function of f, then

$$\text{Domain}(g) = \text{Range}(f) \qquad \text{and} \qquad \text{Range}(g) = \text{Domain}(f).$$

[10] The $\iff$ symbol means that these two statements are equivalent: if one is true, then so is the other.

The defining relationship in **Property 8** is also equivalent to the following two identities, so these provide an alternative characterization of inverse functions:

Property 10. If g is the inverse function of f, then

$$g(f(x)) = x \text{ for every } x \text{ in Domain}(f)$$

and

$$f(g(y)) = y \text{ for every } y \text{ in Domain}(g).$$

Note that the first statement in **Property 10** says that g maps the output $f(x)$ back to the input x. The second statement says the same with the roles of f and g reversed. Therefore, f and g must be inverses.

Property 10 can also be interpreted to say that the functions g and f "undo" each other. If we first apply f to an input x, and then apply g, we get x back again. Likewise, if we apply g to an input y, and then apply f, we get y back again. So whatever action f performs, g reverses it, and vice versa.

I **Example 11.** *Suppose $f(x) = x^3$. Thus, f is the "cubing" function. What operation will reverse the cubing process? Taking a cube root. Thus, the inverse of f should be the function $g(y) = \sqrt[3]{y}$.*

Let's verify **Property 10**:

$$g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$$

and

$$f(g(y)) = f(\sqrt[3]{y}) = (\sqrt[3]{y})^3 = y.$$

I **Example 12.** *Suppose $f(x) = 4x - 1$. f acts on an input x by first multiplying by 4, and then subtracting 1. The inverse function must reverse the process: first add 1, and then divide by 4. Thus, the inverse function should be $g(y) = (y + 1)/4$.*

Again, let's verify **Property 10**:

$$g(f(x)) = g(4x - 1) = \frac{(4x - 1) + 1}{4} = \frac{4x}{4} = x$$

and

$$f(g(y)) = f\left(\frac{y+1}{4}\right) = 4\left(\frac{y+1}{4}\right) - 1 = (y + 1) - 1 = y.$$

Remarks 13.

1. The computation $g(f(x))$, in which the output of one function is used as the input of another, is called the *composition of g with f*. Thus, inverse functions "undo" each other in the sense of composition. Composition of functions is an important concept in many areas of mathematics, so more practice with composition of functions is provided in the exercises.
2. If g is the inverse function of f, then f is also the inverse of g. This follows from either **Property 8** or **Property 10**. (Note that the labels x and y for the variables are unimportant. The key idea is that two functions are inverses if their inputs and outputs are interchanged.)

Notation: In order to indicate that two functions f and g are inverses, we usually use the notation f^{-1} for g. The symbol f^{-1} is read "f inverse". In addition, to avoid confusion with the typical roles of x and y, it's often useful to use different labels for the variables. Rewriting **Property 8** with the f^{-1} notation, and using new labels for the variables, we have the defining relationship:

Property 14.

$$v = f^{-1}(u) \quad \Longleftrightarrow \quad u = f(v)$$

Likewise, rewriting **Property 10**, we have the composition relationships:

Property 15.

$$f^{-1}(f(z)) = z \text{ for every } z \text{ in Domain}(f)$$

and

$$f(f^{-1}(z)) = z \text{ for every } z \text{ in Domain}(f^{-1})$$

However, the new notation comes with an important warning:

Warning 16.

$$f^{-1} \textit{ does } \textbf{not} \textit{ mean } \frac{1}{f}$$

The -1 exponent is just notation in this context. When applied to a function, it stands for the inverse of the function, not the reciprocal of the function.

The Graph of an Inverse Function

How are the graphs of f and f^{-1} related? Suppose that the point (a, b) is on the graph of f. That means that $b = f(a)$. Since inputs and outputs are interchanged for the inverse function, it follows that $a = f^{-1}(b)$, so (b, a) is on the graph of f^{-1}. Now (a, b)

and (b, a) are just reflections of each other across the line $y = x$ (see the discussion below for a detailed explanation), so it follows that the same is true of the graphs of f and f^{-1} if we graph both functions on the same coordinate system (i.e., as functions of x).

For example, consider the functions from **Example 11**. The functions $f(x) = x^3$ and $f^{-1}(x) = \sqrt[3]{x}$ are graphed in **Figure 6** along with the line $y = x$. Several reflected pairs of points are also shown on the graph.

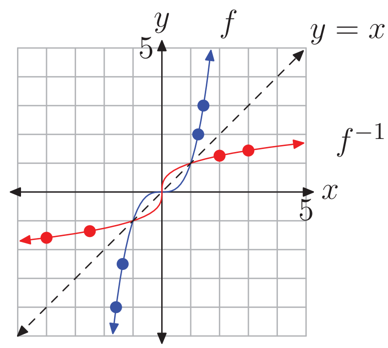


Figure 6. Graphs of $f(x) = x^3$ and $f^{-1}(x) = \sqrt[3]{x}$ are reflections across the line $y = x$.

To see why the points (a, b) and (b, a) are just reflections of each other across the line $y = x$, consider the segment S between these two points (see **Figure 7**). It will be enough to show: (1) that S is perpendicular to the line $y = x$, and (2) that the intersection point P of the segment S and the line $y = x$ is equidistant from each of (a, b) and (b, a).

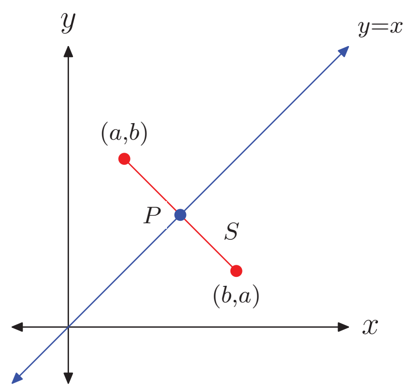


Figure 7. Switching the abscissa and ordinate reflects the point across the line $y = x$.

1. The slope of S is

$$\frac{a-b}{b-a} = -1,$$

and the slope of the line $y = x$ is 1, so they are perpendicular.

2. The line containing S has equation $y - b = -(x - a)$, or equivalently, $y = -x + (a + b)$. To find the intersection of S and the line $y = x$, set $x = -x + (a + b)$ and solve for x to get
$$x = \frac{a+b}{2}.$$
Since $y = x$, it follows that the intersection point is
$$P = \left(\frac{a+b}{2}, \frac{a+b}{2}\right).$$
Finally, we can use the distance formula presented in section 5.6 to compute the distance from P to (a, b) and the distance from P to (b, a). In both cases, the computed distance turns out to be
$$\frac{|a-b|}{\sqrt{2}}.$$

Computing the Formula of an Inverse Function

How does one find the formula of an inverse function? In **Example 11**, it was easy to see that the inverse of the “cubing” function must be the cube root function. But how was the formula for the inverse in **Example 12** obtained?

Actually, there is a simple procedure for finding the formula for the inverse function (provided that such a formula exists; remember that not all functions can be described by a simple formula, so the procedure will not work for such functions). The following procedure works because the inputs and outputs (the x and y variables) are switched in step 3.

Computing the Formula of an Inverse Function

1. Check the graph of the original function $f(x)$ to see if it passes the horizontal line test. If so, then f is one-to-one and you can proceed.
2. Write the formula in xy-equation form, as $y = f(x)$.
3. Interchange the x and y variables.
4. Solve the new equation for y, if possible. The result will be the formula for $f^{-1}(x)$.

Example 17. *Let's start by finding the inverse of the function $f(x) = 4x - 1$ from* **Example 12**.

Step 1: A check of the graph shows that f is one-to-one (see **Figure 8**).

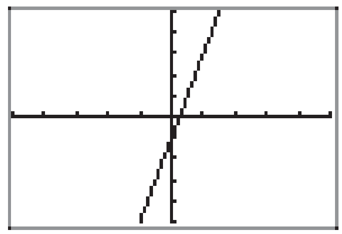


Figure 8. The graph of $f(x) = 4x - 1$ passes the horizontal line test.

Step 2: Write the formula in xy-equation form: $y = 4x - 1$

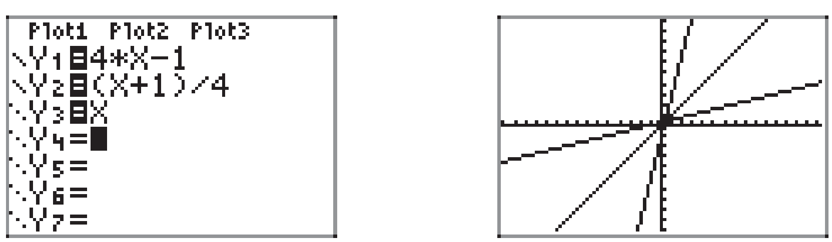
Step 3: Interchange x and y: $x = 4y - 1$

Step 4: Solve for y:

$$\begin{aligned} x &= 4y - 1 \\ \implies x + 1 &= 4y \\ \implies \frac{x+1}{4} &= y \end{aligned}$$

Thus, $f^{-1}(x) = \dfrac{x+1}{4}$.

Figure 9 demonstrates that the graph of $f^{-1}(x) = (x + 1)/4$ is a reflection of the graph of $f(x) = 4x - 1$ across the line $y = x$. In this figure, the `ZSquare` command in the `ZOOM` menu has been used to better illustrate the reflection (the `ZSquare` command equalizes the scales on both axes).



(a) (b)

Figure 9. Symmetry across the line $y = x$

I **Example 18.** *This time we'll find the inverse of* $f(x) = 2x^5 + 3$.

Step 1: A check of the graph shows that f is one-to-one (this is left for the reader to verify).

Step 2: Write the formula in xy-equation form: $y = 2x^5 + 3$

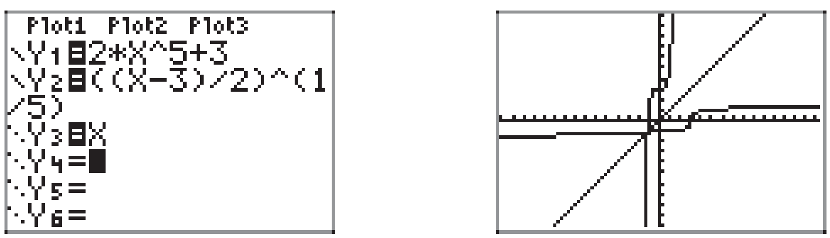
Step 3: Interchange x and y: $x = 2y^5 + 3$

Step 4: Solve for y:

$$\begin{aligned} x &= 2y^5 + 3 \\ \Longrightarrow\ x - 3 &= 2y^5 \\ \Longrightarrow\ \frac{x-3}{2} &= y^5 \\ \Longrightarrow\ \sqrt[5]{\frac{x-3}{2}} &= y \end{aligned}$$

Thus, $f^{-1}(x) = \sqrt[5]{\dfrac{x-3}{2}}$.

Again, note that the graph of $f^{-1}(x) = \sqrt[5]{(x-3)/2}$ is a reflection of the graph of $f(x) = 2x^5 + 3$ across the line $y = x$ (see **Figure 10**).



(a) (b)

Figure 10. Symmetry across the line $y = x$

I **Example 19.** *Find the inverse of* $f(x) = 5/(7 + x)$.

Step 1: A check of the graph shows that f is one-to-one (this is left for the reader to verify).

Step 2: Write the formula in xy-equation form: $y = \dfrac{5}{7+x}$

Step 3: Interchange x and y: $x = \dfrac{5}{7+y}$

Step 4: Solve for y:

$$\begin{aligned} x &= \frac{5}{7+y} \\ \implies x(7+y) &= 5 \\ \implies 7+y &= \frac{5}{x} \\ \implies y &= \frac{5}{x} - 7 = \frac{5-7x}{x} \end{aligned}$$

Thus, $f^{-1}(x) = \dfrac{5-7x}{x}$.

I **Example 20.** *This example is a bit more complicated: find the inverse of the function* $f(x) = (5x+2)/(x-3)$.

Step 1: A check of the graph shows that f is one-to-one (this is left for the reader to verify).

Step 2: Write the formula in xy-equation form: $y = \dfrac{5x+2}{x-3}$

Step 3: Interchange x and y: $x = \dfrac{5y+2}{y-3}$

Step 4: Solve for y:

$$\begin{aligned} x &= \frac{5y+2}{y-3} \\ \implies x(y-3) &= 5y+2 \\ \implies xy - 3x &= 5y+2 \end{aligned}$$

This equation is linear in y. Isolate the terms containing the variable y on one side of the equation, factor, then divide by the coefficient of y.

$$\begin{aligned} xy - 3x &= 5y+2 \\ \implies xy - 5y &= 3x+2 \\ \implies y(x-5) &= 3x+2 \\ \implies y &= \frac{3x+2}{x-5} \end{aligned}$$

Thus, $f^{-1}(x) = \dfrac{3x+2}{x-5}$.

I **Example 21.** *According to the horizontal line test, the function* $h(x) = x^2$ *is certainly not one-to-one. However, if we only consider the right half or left half of the function (i.e., restrict the domain to either the interval* $[0, \infty)$ *or* $(-\infty, 0]$*), then* *the function* would *be one-to-one, and therefore would have an inverse* (**Figure 11**(*a*) *shows the left half*)*. For example, suppose f is the function*

$$f(x) = x^2, \quad x \leq 0.$$

In this case, the procedure still works, provided that we carry along the domain condition in all of the steps, as follows:

Step 1: The graph in **Figure 11**(a) passes the horizontal line test, so f is one-to-one.

Step 2: Write the formula in xy-equation form:

$$y = x^2, \quad x \leq 0$$

Step 3: Interchange x and y:

$$x = y^2, \quad y \leq 0$$

Note how x and y must also be interchanged in the domain condition.

Step 4: Solve for y:

$$y = \pm\sqrt{x}, \quad y \leq 0$$

Now there are two choices for y, one positive and one negative, but the condition $y \leq 0$ tells us that the negative choice is the correct one. Thus, the last statement is equivalent to

$$y = -\sqrt{x}.$$

Thus, $f^{-1}(x) = -\sqrt{x}$. The graph of f^{-1} is shown in **Figure 11**(b), and the graphs of both f and f^{-1} are shown in **Figure 11**(c) as reflections across the line $y = x$.

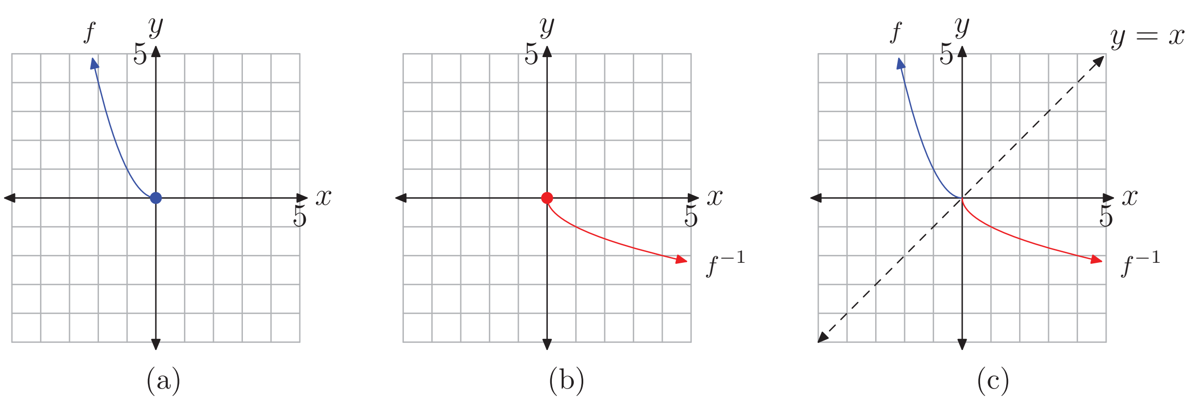


Figure 11.

4.5 *Exercises*

In **Exercises 1-12**, use the graph to determine whether the function is one-to-one.

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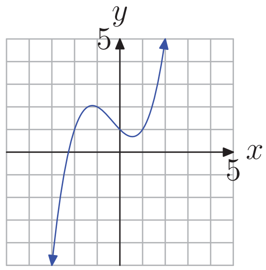

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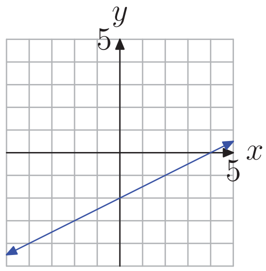

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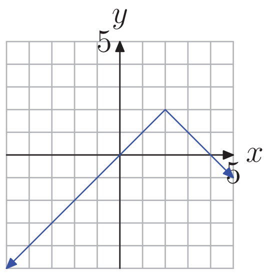

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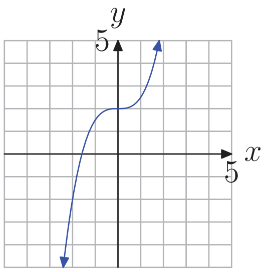

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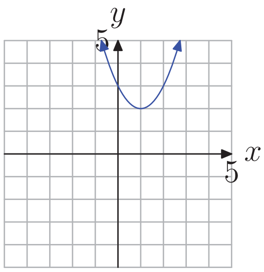

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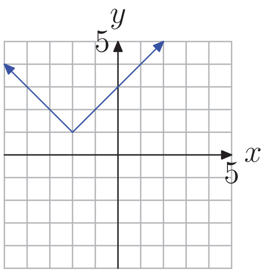

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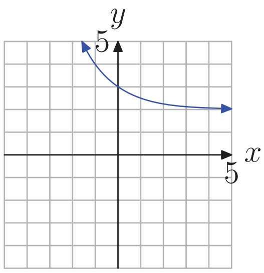

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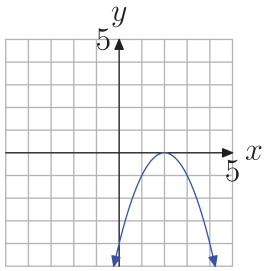

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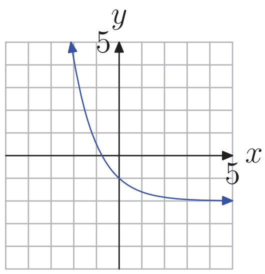

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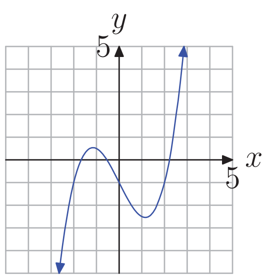

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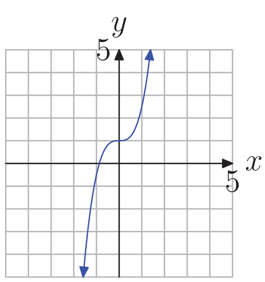

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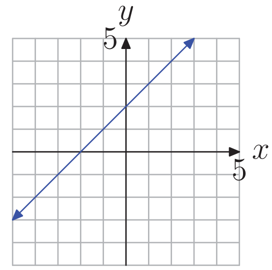


In **Exercises 13-28**, evaluate the composition $g(f(x))$ and simplify your answer.

13. $g(x) = \frac{9}{x}$, $f(x) = -2x^2 + 5x - 2$

14. $f(x) = -\frac{5}{x}$, $g(x) = -4x^2 + x - 1$

15. $g(x) = 2\sqrt{x}$, $f(x) = -x - 3$

16. $f(x) = 3x^2 - 3x - 5$, $g(x) = \frac{6}{x}$

17. $g(x) = 3\sqrt{x}$, $f(x) = 4x + 1$

18. $f(x) = -3x - 5$, $g(x) = -x - 2$

19. $g(x) = -5x^2 + 3x - 4$, $f(x) = \frac{5}{x}$

20. $g(x) = 3x + 3$, $f(x) = 4x^2 - 2x - 2$

21. $g(x) = 6\sqrt{x}$, $f(x) = -4x + 4$

22. $g(x) = 5x - 3$, $f(x) = -2x - 4$

23. $g(x) = 3\sqrt{x}$, $f(x) = -2x + 1$

24. $g(x) = \frac{3}{x}$, $f(x) = -5x^2 - 5x - 4$

25. $f(x) = \frac{5}{x}$, $g(x) = -x + 1$

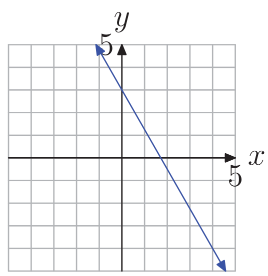
26. $f(x) = 4x^2 + 3x - 4$, $g(x) = \frac{2}{x}$

27. $g(x) = -5x + 1,\ f(x) = -3x - 2$

28. $g(x) = 3x^2 + 4x - 3,\ f(x) = \dfrac{8}{x}$

In **Exercises 29-36**, first copy the given graph of the one-to-one function $f(x)$ onto your graph paper. Then on the same coordinate system, sketch the graph of the inverse function $f^{-1}(x)$.

29.



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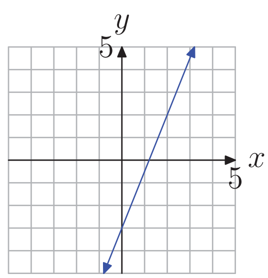

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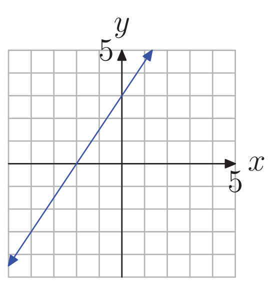

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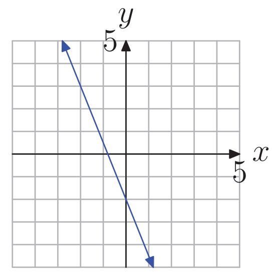

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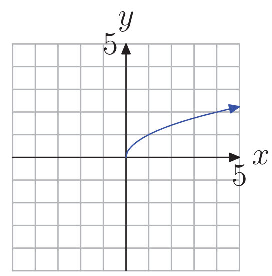

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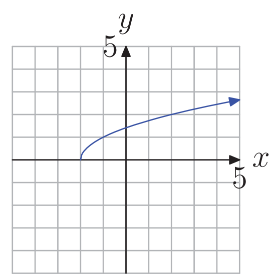

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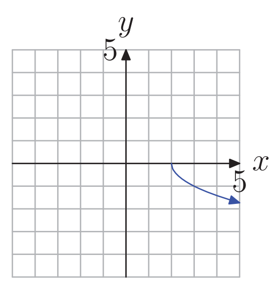

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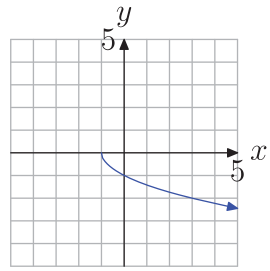


In **Exercises 37-68**, find the formula for the inverse function $f^{-1}(x)$.

37. $f(x) = 5x^3 - 5$

38. $f(x) = 4x^7 - 3$

39. $f(x) = -\dfrac{9x-3}{7x+6}$

40. $f(x) = 6x - 4$

41. $f(x) = 7x - 9$

42. $f(x) = 7x + 4$

43. $f(x) = 3x^5 - 9$

44. $f(x) = 6x + 7$

45. $f(x) = \dfrac{4x+2}{4x+3}$

46. $f(x) = 5x^7 + 4$

47. $f(x) = \dfrac{4x-1}{2x+2}$

48. $f(x) = \sqrt[7]{8x-3}$

49. $f(x) = \sqrt[3]{-6x-4}$

50. $f(x) = \dfrac{8x-7}{3x-6}$

51. $f(x) = \sqrt[7]{-3x-5}$

52. $f(x) = \sqrt[9]{8x+2}$

53. $f(x) = \sqrt[3]{6x+7}$

54. $f(x) = \dfrac{3x+7}{2x+8}$

55. $f(x) = -5x + 2$

56. $f(x) = 6x + 8$

57. $f(x) = 9x^9 + 5$

58. $f(x) = 4x^5 - 9$

59. $f(x) = \dfrac{9x-3}{9x+7}$

60. $f(x) = \sqrt[3]{9x-7}$

61. $f(x) = x^4,\ x \le 0$

62. $f(x) = x^4,\ x \ge 0$

63. $f(x) = x^2 - 1,\ x \le 0$

64. $f(x) = x^2 + 2,\ x \ge 0$

65. $f(x) = x^4 + 3,\ x \le 0$

66. $f(x) = x^4 - 5,\ x \ge 0$

67. $f(x) = (x-1)^2,\ x \le 1$

68. $f(x) = (x+2)^2,\ x \ge -2$

4.5 *Answers*

1. not one-to-one

3. not one-to-one

5. not one-to-one

7. one-to-one

9. one-to-one

11. one-to-one

13. $-\dfrac{9}{2x^2-5x+2}$

15. $2\sqrt{-x-3}$

17. $3\sqrt{4x+1}$

19. $-\dfrac{125}{x^2}+\dfrac{15}{x}-4$

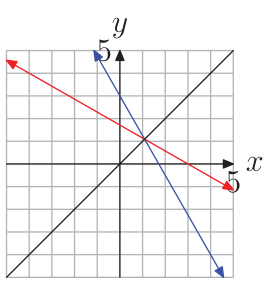
21. $6\sqrt{-4x+4}$

23. $3\sqrt{-2x+1}$

25. $-5/x+1$

27. $15x+11$

29.



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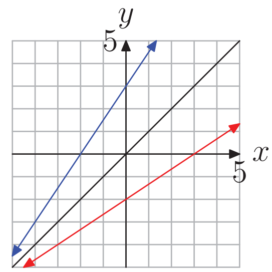

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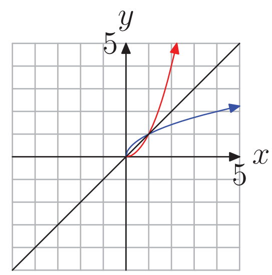

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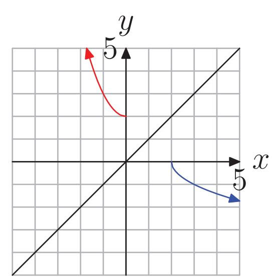


37. $\sqrt[3]{\dfrac{x+5}{5}}$

39. $-\dfrac{6x-3}{7x+9}$

41. $\dfrac{x+9}{7}$

43. $\sqrt[5]{\dfrac{x+9}{3}}$

45. $-\dfrac{3x-2}{4x-4}$

47. $-\dfrac{2x+1}{2x-4}$

49. $-\dfrac{x^3+4}{6}$

51. $-\dfrac{x^7+5}{3}$

53. $\dfrac{x^3-7}{6}$

55. $-\dfrac{x-2}{5}$

57. $\sqrt[9]{\dfrac{x-5}{9}}$

59. $-\dfrac{7x+3}{9x-9}$

61. $-\sqrt[4]{x}$

63. $-\sqrt{x+1}$

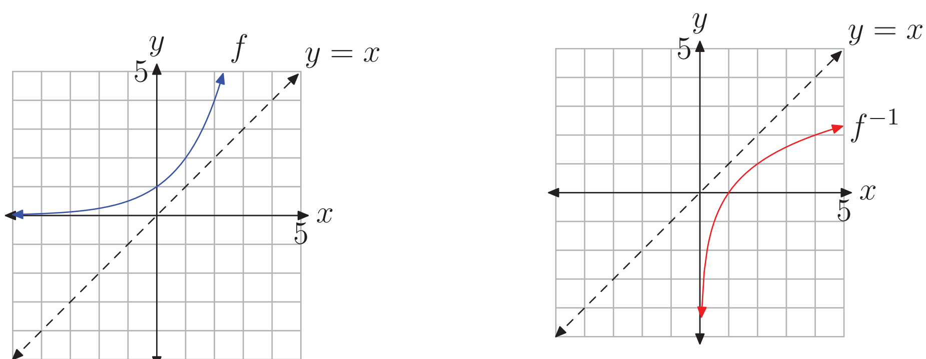
65. $-\sqrt[4]{x-3}$

67. $-\sqrt{x}+1$

4.6 Logarithmic Functions

We can now apply the inverse function theory from the previous section to the exponential function. From Section 8.2, we know that the function $f(x) = b^x$ is either increasing (if $b > 1$) or decreasing (if $0 < b < 1$), and therefore is one-to-one. Consequently, f has an inverse function f^{-1}.

As an example, let's consider the exponential function $f(x) = 2^x$. f is increasing, has domain $D_f = (-\infty, \infty)$, and range $R_f = (0, \infty)$. Its graph is shown in **Figure 1**(a). The graph of the inverse function f^{-1} is a reflection of the graph of f across the line $y = x$, and is shown in **Figure 1**(b). Since domains and ranges are interchanged, the domain of the inverse function is $D_{f^{-1}} = (0, \infty)$ and the range is $R_{f^{-1}} = (-\infty, \infty)$.



(a) (b)

Figure 1. The graphs of $f(x) = 2^x$ and its inverse $f^{-1}(x)$ are reflections across the line $y = x$.

Unfortunately, when we try to use the procedure given in Section 8.4 to find a formula for f^{-1}, we run into a problem. Starting with $y = 2^x$, we then interchange x and y to obtain $x = 2^y$. But now we have no algebraic method for solving this last equation for y. It follows that the inverse of $f(x) = 2^x$ has no formula involving the usual arithmetic operations and functions that we're familiar with. Thus, the inverse function is a new function. The name of this new function is the *logarithm of x to base 2*, and it's denoted by $f^{-1}(x) = \log_2(x)$.

Recall that the defining relationship between a function and its inverse (Property 14 in Section 8.4) simply states that the inputs and outputs of the two functions are interchanged. Thus, the relationship between 2^x and its inverse $\log_2(x)$ takes the following form:

$$v = \log_2(u) \quad \iff \quad u = 2^v$$

More generally, for each exponential function $f(x) = b^x$ ($b > 0$, $b \neq 1$), the inverse function $f^{-1}(x)$ is called the *logarithm of x to base b*, and is denoted by $\log_b(x)$. The defining relationship is given in the following definition.

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Definition 1. *If $b > 0$ and $b \neq 1$, then the* **logarithm of u to base b** *is defined by the relationship*

$$v = \log_b(u) \quad \Longleftrightarrow \quad u = b^v. \tag{2}$$

In order to understand the logarithm function better, let's work through a few simple examples.

I **Example 3.** *Compute* $\log_2(8)$.

Label the required value by v, so $v = \log_2(8)$. Then by (**2**), using $b = 2$ and $u = 8$, it follows that $2^v = 8$, and therefore $v = 3$ (solving by inspection).

In the last example, note that $\log_2(8) = 3$ is the exponent v such that $2^v = 8$. Thus, in general, one way to interpret the definition of the logarithm in (**2**) is that $\log_b(u)$ is the exponent v such that $b^v = u$. In other words, the value of the logarithm is the exponent!

I **Example 4.** *Compute* $\log_{10}(10\,000)$.

Again, label the required value by v, so $v = \log_{10}(10\,000)$. By (**2**), it follows that $10^v = 10\,000$, and therefore $v = 4$. Note that here again we have found the exponent v=4 that is needed for base 10 in order to get $10^v = 10\,000$.

I **Example 5.** *Compute* $\log_3\left(\frac{1}{9}\right)$.

$$\begin{aligned} & v = \log_3\left(\frac{1}{9}\right) \\ \Longrightarrow\ & 3^v = \frac{1}{9} && \text{by } (\mathbf{2}) \\ \Longrightarrow\ & v = -2 && \text{since } 3^{-2} = \frac{1}{9} \end{aligned}$$

I **Example 6.** *Solve the equation* $\log_5(x) = 1$.

$$\begin{aligned} & \log_5(x) = 1 \\ \Longrightarrow\ & 5^1 = x && \text{by } (\mathbf{2}) \\ \Longrightarrow\ & x = 5 \end{aligned}$$

I **Example 7.** *Solve the equation* $\log_b(64) = 3$ *for* b.

$$\begin{aligned} &\log_b(64) = 3 \\ \Longrightarrow\ & b^3 = 64 \qquad \text{by } (2) \\ \Longrightarrow\ & b = \sqrt[3]{64} = 4 \end{aligned}$$

I **Example 8.** *Solve the equation* $\log_{1/2}(x) = -2$.

$$\begin{aligned} &\log_{1/2}(x) = -2 \\ \Longrightarrow\ & \left(\frac{1}{2}\right)^{-2} = x \qquad \text{by } (2) \\ \Longrightarrow\ & x = \frac{1}{\left(\frac{1}{2}\right)^2} = \frac{1}{\frac{1}{4}} = 4 \end{aligned}$$

The composition relationships in Property 15 of Section 8.4, applied to b^x and $\log_b(x)$, become

Property 9.

$$\log_b(b^x) = x \tag{10}$$

and

$$b^{\log_b(x)} = x. \tag{11}$$

Both equations are important. Note that (**11**) again shows that the $\log_b(x)$ is the exponent v such that $b^v = x$. Equation (**10**) will be used frequently in this and later sections to help us solve exponential equations.

Logarithmic functions are used in many areas of science and engineering. For example, they are used to define the Richter scale for the magnitudes of earthquakes, the decibel scale for the loudness of sound, and the astronomical scale for stellar brightness. They are also important tools for use in computation (as we will see in Section 8.8). Our main use of logarithms in this textbook will be to solve exponential equations, and thereby help us study physical phenomena that are described by exponential functions (as in Section 8.7).

Computing Logarithms

In Examples **3**–**8** above, we were able to compute the logarithms by converting to exponential equations that could be solved by inspection. But it's easy to see that most of the time this won't work. For example, how would we compute the value of $\log_2(7)$?

Fortunately, mathematicians have found other methods for computing logarithms to high accuracy, and they can now be easily approximated using a calculator or computer.

Your calculator has built-in buttons for computing two different logarithms, $\log_{10}(x)$ and $\log_e(x)$. $\log_{10}(x)$ is called the *common logarithm*, and $\log_e(x)$ is called the *natural logarithm*.

Common Logarithm: The common logarithm $\log_{10}(x)$ is computed using the `LOG` button on your calculator. Notice also that its inverse function 10^x, can be computed using the same button in conjunction with the `2ND` button. The common logarithm is usually the most convenient one to use for computations involving scientific notation (because we use a base 10 number system), and therefore is the logarithm most often used in the physical sciences. Because of that, it's often just abbreviated by $\log(x)$, and we'll do that as well in the remainder of the text.

Common Logarithm. $\log(x)$ and $\log_{10}(x)$ are equivalent notations. Thus, we have the defining relationship

$$v = \log(u) \quad \iff \quad u = 10^v.$$

The composition properties for the common logarithm are

$$\log(10^x) = x \tag{12}$$

and

$$10^{\log(x)} = x.$$

Natural Logarithm: The natural logarithm $\log_e(x)$ is computed using the `LN` button on your calculator. Its inverse function, e^x, is computed using the same button in conjunction with the `2ND` button. The natural logarithm turns out to be the most convenient one to use in mathematics, because a lot of formulas, especially in calculus, are much simpler when the natural logarithm is used. The natural logarithm is abbreviated by $\ln(x)$.

Natural Logarithm. $\ln(x)$ and $\log_e(x)$ are equivalent notations. Thus, we have the defining relationship

$$v = \ln(u) \quad \iff \quad u = e^v.$$

The composition properties for the common logarithm are

$$\ln(e^x) = x \tag{13}$$

and

$$e^{\ln(x)} = x.$$

Note that when using your calculator to compute $\log(x)$ and $\ln(x)$, you will usually only obtain approximate values, as these values frequently are irrational numbers.

What about other bases? You can also compute these on your calculator, but we'll first need to develop the *Change of Base Formula* in the next section. However, at this point, we can at least solve exponential equations involving bases 10 and e, as shown in the next two examples.

I **Example 14.** *Solve the equation* $704 = 2(10)^x$.

The first step is to isolate the exponential on the right side by dividing both sides by 2:

$$352 = 10^x$$

Then simply apply the $\log_{10}(x)$ function to both sides of the equation:

$$\log_{10}(352) = \log_{10}(10^x)$$

But (**10**) implies that $\log_{10}(10^x) = x$. Therefore, $x = \log_{10}(352) = \log(352)$ is the exact solution. The approximate value, using a calculator, is 2.546542663 (see **Figure 2**).

Alternatively, instead of taking the logarithm of both sides in the second step, you can apply (**2**) to the equation $352 = 10^x$ to get $x = \log_{10}(352)$.

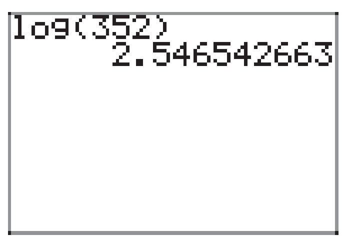


Figure 2. Approximation of $\log(352) = \log_{10}(352)$.

This last example shows how logarithms can be used for solving exponential equations. The basic strategy is to first isolate the exponential on one side of the equation, and then take appropriate logarithms of both sides. Here's one more example for now, and then we'll return to this process repeatedly in the remaining sections, especially when we work with application problems.

I **Example 15.** *Solve the equation* $30 = 20e^x$.

First isolate the exponential on the right side by dividing both sides by 20:

$$1.5 = e^x$$

This time, since the base of the exponential function is e, apply the natural logarithm function to both sides:

$$\log_e(1.5) = \log_e(e^x)$$

Simplify the right side, since $\log_e(e^x) = x$ by (**10**):

$$\log_e(1.5) = x$$

Therefore, $x = \log_e(1.5) = \ln(1.5)$ is the exact solution. The approximate value, using a calculator, is 0.4054651081 (see **Figure 3**).

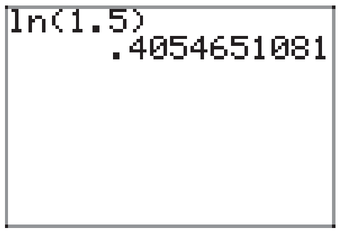


Figure 3. Approximation of $\ln 1.5 = \log_e(1.5)$.

In the next section, we'll learn how to solve exponential equations involving other bases.

Graphs of Logarithmic Functions

At the beginning of this section, we looked at the graphs of $f(x) = 2^x$ and its inverse function $f^{-1}(x) = \log_2(x)$. More generally, the graph of the exponential function $f(x) = b^x$ for $b > 1$ is shown in **Figure 4**(a), along with its inverse logarithmic function $f^{-1}(x) = \log_b(x)$. According to Section 8.4, the two graphs are reflections across the line $y = x$. Similarly, the graph for $0 < b < 1$ is shown in **Figure 4**(b).

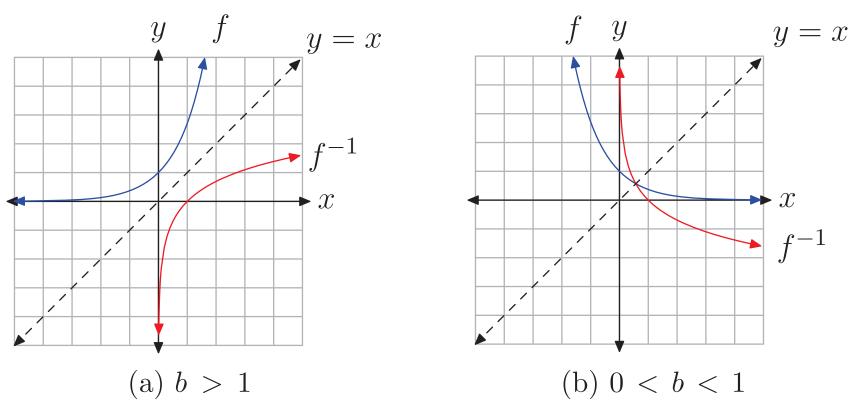


Figure 4. The graphs of $f(x) = b^x$ and $f^{-1}(x) = \log_b(x)$ are reflections across the line $y = x$.

Because domains and ranges of inverse functions are interchanged, it follows that

Property 16.

$$\text{Domain}(\log_b(x)) = (0, \infty)$$

and

$$\text{Range}(\log_b(x)) = (-\infty, \infty).$$

In particular, note that the logarithm of a negative number, as well as the logarithm of 0, are *not defined*.

Two particular points on the graph of the logarithm are noteworthy. Since $b^0 = 1$, it follows that $\log_b(1) = 0$, and therefore the x-intercept of the graph of $\log_b(x)$ is $(1, 0)$. Similarly, since $b^1 = b$, it follows that $\log_b(b) = 1$, and therefore $(b, 1)$ is on the graph.

Property 17.

$$\log_b(1) = 0 \qquad \textit{and} \qquad \log_b(b) = 1$$

Finally, since the graph of b^x has a horizontal asymptote $y = 0$, the graph of $\log_b(x)$ must have a vertical asymptote $x = 0$. This behavior is a consequence of the fact that inputs and outputs of inverse functions are interchanged, and can be observed in **Figure 4**.

In the final example below, we'll apply a transformation to the logarithm and see how that affects the graph.

I **Example 18.** *Plot the graph of the function* $f(x) = \log_2(x + 1)$.

The graph of $f(x) = \log_2(x+1)$ will be the same as the graph of $g(x) = \log_2(x)$ shifted one unit to the left. The graph of g is shown in **Figure 1**(b). The x-intercept $(1, 0)$ on the graph of g will be shifted one unit to the left to $(0, 0)$ on the graph of f. Likewise, the vertical asymptote $x = 0$ on the graph of g will be shifted one unit to the left to the line $x = -1$ on the graph of f. The final graph of f is shown in **Figure 5**.

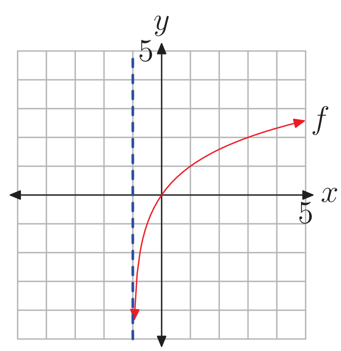


Figure 5. The graph of $f(x) = \log_2(x + 1)$.

4.6 Exercises

In **Exercises 1-18**, find the exact value of the function at the given value b.

1. $f(x) = \log_3(x)$; $b = \sqrt[5]{3}$.

2. $f(x) = \log_5(x)$; $b = 3125$.

3. $f(x) = \log_2(x)$; $b = \frac{1}{16}$.

4. $f(x) = \log_2(x)$; $b = 4$.

5. $f(x) = \log_5(x)$; $b = 5$.

6. $f(x) = \log_2(x)$; $b = 8$.

7. $f(x) = \log_2(x)$; $b = 32$.

8. $f(x) = \log_4(x)$; $b = \frac{1}{16}$.

9. $f(x) = \log_5(x)$; $b = \frac{1}{3125}$.

10. $f(x) = \log_5(x)$; $b = \frac{1}{25}$.

11. $f(x) = \log_5(x)$; $b = \sqrt[6]{5}$.

12. $f(x) = \log_3(x)$; $b = \sqrt[3]{3}$.

13. $f(x) = \log_6(x)$; $b = \sqrt[6]{6}$.

14. $f(x) = \log_5(x)$; $b = \sqrt[5]{5}$.

15. $f(x) = \log_2(x)$; $b = \sqrt[6]{2}$.

16. $f(x) = \log_4(x)$; $b = \frac{1}{4}$.

17. $f(x) = \log_3(x)$; $b = \frac{1}{9}$.

18. $f(x) = \log_4(x)$; $b = 64$.

In **Exercises 19-26**, use a calculator to evaluate the function at the given value p. Round your answer to the nearest hundredth.

19. $f(x) = \ln(x)$; $p = 10.06$.

20. $f(x) = \ln(x)$; $p = 9.87$.

21. $f(x) = \ln(x)$; $p = 2.40$.

22. $f(x) = \ln(x)$; $p = 9.30$.

23. $f(x) = \log(x)$; $p = 7.68$.

24. $f(x) = \log(x)$; $p = 652.22$.

25. $f(x) = \log(x)$; $p = 6.47$.

26. $f(x) = \log(x)$; $p = 86.19$.

In **Exercises 27-34**, solve the given equation, and round your answer to the nearest hundredth.

27. $13 = e^{8x}$

28. $2 = 8e^{x}$

29. $19 = 10^{8x}$

30. $17 = 10^{2x}$

31. $7 = 6(10)^{x}$

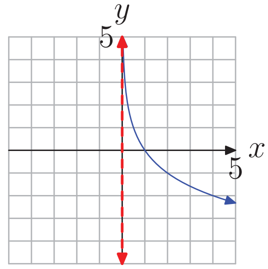
32. $7 = e^{9x}$

33. $13 = 8e^{x}$

34. $5 = 7(10)^{x}$

In **Exercises 35-42**, the graph of a logarithmic function of the form $f(x) = \log_b(x - a)$ is shown. The dashed red line is a vertical asymptote. Determine the domain of the function. Express your answer in interval notation.

35.



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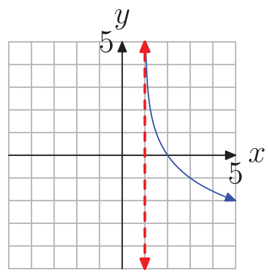

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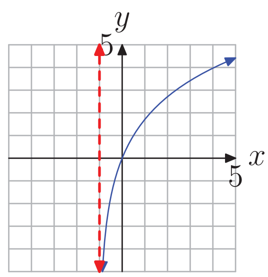

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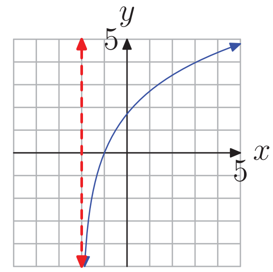

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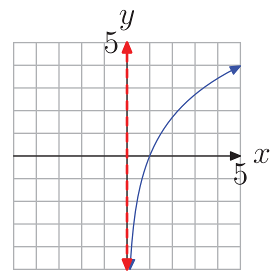

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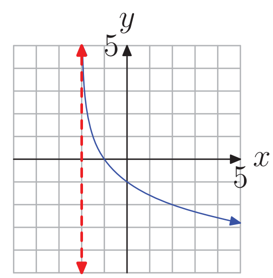

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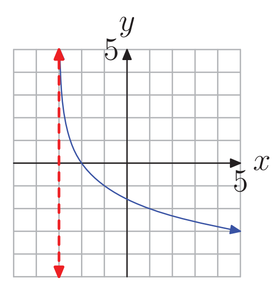

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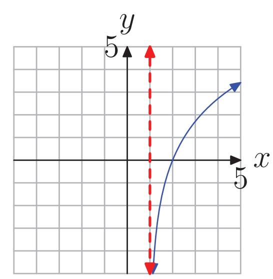


4.6 *Answers*

1. $\frac{1}{5}$

3. -4

5. 1

7. 5

9. -5

11. $\frac{1}{6}$

13. $\frac{1}{6}$

15. $\frac{1}{6}$

17. -2

19. 2.31

21. 0.88

23. 0.89

25. 0.81

27. 0.32

29. 0.16

31. 0.07

33. 0.49

35. $(0, \infty)$

37. $(-1, \infty)$

39. $(0, \infty)$

41. $(-3, \infty)$

4.7 Properties of Logarithms; Solving Exponential Equations

Logarithms were actually discovered and used in ancient times by both Indian and Islamic mathematicians. They were not used widely, though, until the 1600âĂŹs, when logarithms simplified the large amounts of hand computations needed in the scientific explorations of the times. In particular, after the invention of the telescope, calculations involving astronomical data became very important, and logarithms became an essential mathematical tool. Indeed, until the invention of the computer and electronic calculator in recent times, hand calculations using logarithms were a staple of every science student's curriculum.

The usefulness of logarithms in calculations is based on the following three important properties, known generally as the *properties of logarithms*.

Properties of Logarithms

a) $\log_b(MN) = \log_b(M) + \log_b(N)$

b) $\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$

c) $\log_b(M^r) = r\log_b(M)$

provided that $M, N, b > 0$.

The first property says that the "log of a product is the sum of the logs." The second says that the "log of a quotient is the difference of the logs." And the third property is sometimes referred to as the "power rule". Loosely speaking, when taking the log of a power, you can just move the exponent out in front of the log.

We won't go into the details of the computation procedures using properties (a) and (b), since these procedures are no longer necessary after the invention of the calculator. But the idea is that a time-consuming product of two numbers, for example two 10-digit numbers, can be transformed by property (a) into a much simpler addition problem. Similarly, a large and difficult quotient can be transformed by property (b) into a much simpler subtraction problem. Properties (a) and (b) are also the basis for the slide rule, a mechanical computation device that preceded the electronic calculator (very fast and useful, but only accurate to about three digits).

Property (c), on the other hand, is still useful for difficult computations. If you try to compute a large power, say 2^{100}, on a calculator or computer, you'll get an error message. That's because all calculators and computers can only handle numbers and exponents within a certain range. So to compute a large power, it's necessary to use property (c) to turn it into a multiplication problem. The details of this procedure are given in Section 8.8.

Even though properties (a) and (b) are no longer necessary for computation purposes, that does not mean they are not important. Logarithmic functions serve many purposes in mathematics and the sciences, and all of the logarithm properties are useful in various ways.

Where do the logarithm properties come from? Actually, they're all derived from the laws of exponents, using the fact that the exponential function is the inverse of the logarithm function. Since we'll only be using property (c) in this book, we'll show how that property is derived. Properties (a) and (b) are derived in a similar manner.

Proof of (c): Start on the right side of the equation, and label $\log_b(M)$ by x:

$$x = \log_b(M)$$

Use Definition 1 in Section 8.5 to rewrite the equation in exponential form:

$$b^x = M$$

Raise both sides to the rth power:

$$(b^x)^r = M^r$$

Apply one of the Laws of Exponents to the left side:

$$b^{rx} = M^r$$

Apply the base b logarithmic function to both sides:

$$\log_b(b^{rx}) = \log_b(M^r)$$

Apply formula (10) in Section 8.5 to the left side:

$$rx = \log_b(M^r)$$

Substitute back for x from the first line above:

$$r\log_b(M) = \log_b(M^r)$$

This is the formula in property (c).

Change of Base Formula

We can now prove a conversion formula that will enable us to compute the logarithm to any base.

Change of Base Formula:

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

Proof: Start on the left side of the equation, and label $\log_a(x)$ by r:

$$r = \log_a(x)$$

Use Definition 1 in Section 8.5 to rewrite the equation in exponential form:

$$a^r = x$$

Apply the base b logarithmic function to both sides:

$$\log_b(a^r) = \log_b(x)$$

Apply property (c) to the left side:

$$r \log_b(a) = \log_b(x)$$

Divide by $\log_b(a)$:

$$r = \frac{\log_b(x)}{\log_b(a)}$$

Substitute back for r from the first line above:

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

This is the Change of Base Formula.

I **Example 1.** *Compute* $\log_2(5)$.

Before applying the Change of Base Formula, let's see if we can estimate the value of $\log_2(5)$. First recall from Property 9 in Section 8.5 that $2^{\log_2(5)} = 5$. Now how large would the exponent on a base of 2 need to be for the power to equal 5? Since $2^2 = 4$ (too small) and $2^3 = 8$ (too large), we should expect $\log_2(5)$ to lie somewhere between 2 and 3. Indeed, applying the Change of Base Formula with the common logarithm yields

$$\log_2(5) = \frac{\log_{10}(5)}{\log_{10}(2)} = \frac{\log(5)}{\log(2)} \approx \frac{.6989700043}{.3010299957} \approx 2.321928095.$$

According to the formula, we could instead use the natural logarithm to obtain the same answer, as in

$$\log_2(5) = \frac{\log_e(5)}{\log_e(2)} = \frac{\ln(5)}{\ln(2)} \approx \frac{1.609437912}{.6931471806} \approx 2.321928095.$$

Calculator keystrokes are shown in **Figure 1**.

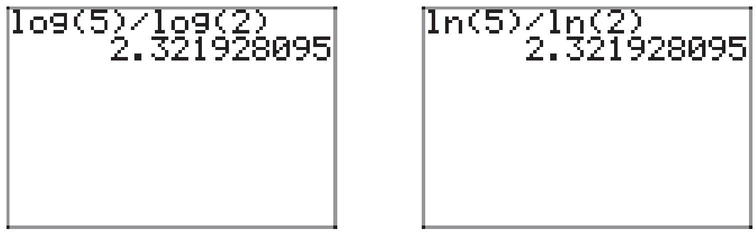


Figure 1. Computing $\log_2(5)$ using the Change of Base Formula.

Another way to view the Change of Base Formula is that it says that *all logarithms are multiples of each other*, since

$$\log_a(x) = \left(\frac{1}{\log_b(a)}\right)\log_b(x).$$

Thus, $\log_a(x)$ is a constant multiple of $\log_b(x)$, where the constant is $1/\log_b(a)$.

Solving Exponential Equations

Property (c) ($\log_b(M^r) = r\log_b(M)$) is also used extensively to help solve exponential equations, and thus will be an important tool when we work with applications in the next section. In general terms, the main strategy for solving exponential equations is to (1) first isolate the exponential, then (2) apply a logarithmic function to both sides, and then (3) use property (c). We'll illustrate the strategy with several examples.

I **Example 2.** *Solve* $8 = 5(3^x)$.

Before trying the procedure outlined above, let's first approximate the solution using a graphical approach. Graph both sides of the equation in your calculator, and then find the intersection of the two curves to obtain $x \approx 0.42781574$ (see **Figure 2**).

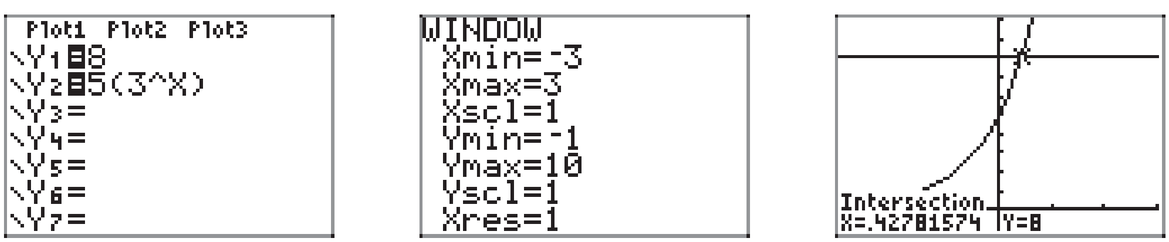


Figure 2. Approximating the solution of $8 = 5(3^x)$ graphically.

Now we'll solve the equation algebraically. First isolate the exponential function on one side of the equation by dividing both sides by 5:

$$1.6 = 3^x$$

Then take the logarithm of both sides. Use either the common or natural log:

$$\log(1.6) = \log(3^x)$$

Now use property (c) to move the exponent in front of the log on the right side:

$$\log(1.6) = x\log(3)$$

Finally, solve for x by dividing both sides by $\log(3)$:

$$\frac{\log(1.6)}{\log(3)} = x$$

Thus, the exact value of x is $\dfrac{\log(1.6)}{\log(3)}$, and the approximate value is 0.42781574. Note that this is the same as the graphical approximation found earlier.

I **Example 3.** *Solve* $300 = 100(1.05^{5x})$.

$$\begin{aligned}
& 300 = 100(1.05^{5x}) & \\
\Longrightarrow\ & 3 = 1.05^{5x} & \text{isolate the exponential} \\
\Longrightarrow\ & \log(3) = \log(1.05^{5x}) & \text{apply the common log function} \\
\Longrightarrow\ & \log(3) = 5x\log(1.05) & \text{use property (c)} \\
\Longrightarrow\ & \frac{\log(3)}{5\log(1.05)} = x & \text{divide} \\
\Longrightarrow\ & x \approx 4.503417061 &
\end{aligned}$$

If the base of the exponential is either 10 or e, the correct choice of logarithm leads to a faster solution:

I **Example 4.** *Solve* $3 = 4e^x$.

$$\begin{aligned}
& 3 = 4e^x & \\
\Longrightarrow\ & 0.75 = e^x & \text{isolate the exponential} \\
\Longrightarrow\ & \ln(0.75) = \ln(e^x) & \text{apply the natural log function} \\
\Longrightarrow\ & \ln(0.75) = x & \text{since } \ln(e^x) = x \\
\Longrightarrow\ & x \approx -.2876820725 &
\end{aligned}$$

In this case, because the base of the exponential function is e, the use of the natural log function simplifies the solution.

We can now turn our attention to solving more interesting application problems, such as the questions raised at the end of Section 8.3.

I **Example 5.** *If you deposit \$1000 in an account paying 6% interest compounded continuously, how long will it take for you to have \$1500 in your account?*

First, recall the continuous compound interest formula from Section 8.3:

$$P(t) = P_0 e^{rt} \tag{6}$$

In this case, $P_0 = 1000$ and $r = .06$. Inserting these values into the formula, we obtain

$$P(t) = 1000e^{0.06t}.$$

Now we want the future value $P(t)$ of the account at some time t to equal \$1500. Therefore, we must solve the equation

$$1500 = 1000e^{0.06t}.$$

Following the steps in the previous example,

$$\begin{aligned}
& 1500 = 1000e^{0.06t} && \\
\implies\ & 1.5 = e^{0.06t} && \text{isolate the exponential} \\
\implies\ & \ln(1.5) = \ln(e^{0.06t}) && \text{apply the natural log function} \\
\implies\ & \ln(1.5) = 0.06t && \text{since } \ln(e^x) = x \\
\implies\ & \frac{\ln(1.5)}{0.06} = t && \text{divide} \\
\implies\ & t \approx 6.757751802. &&
\end{aligned}$$

Thus, it would take about 6 years and 9 months.

I **Example 7.** *If you deposit \$1000 in an account paying 5% interest compounded monthly, how long will it take for your money to double?*

First, recall the discrete compound interest formula from Section 8.3:

$$P(t) = P_0 \left(1 + \frac{r}{n}\right)^{nt} \tag{8}$$

In this case, $P_0 = 1000$, $r = .05$, and $n = 12$. Inserting these values into the formula, we obtain

$$P(t) = 1000 \left(1 + \frac{.05}{12}\right)^{12t}.$$

Now we want the future value $P(t)$ of the account at some time t to equal twice the initial amount. In other words, we want $P(t)$ to equal 2000. Therefore, we must solve the equation

$$2000 = 1000 \left(1 + \frac{.05}{12}\right)^{12t}.$$

Following the steps in Examples **2** and **3**,

$$
\begin{aligned}
2000 &= 1000\left(1+\frac{.05}{12}\right)^{12t} && \\
\implies 2 &= \left(1+\frac{.05}{12}\right)^{12t} && \text{isolate the exponential}\\
\implies \log(2) &= \log\left(\left(1+\frac{.05}{12}\right)^{12t}\right) && \text{apply the common log function}\\
\implies \log(2) &= 12t\log\left(1+\frac{.05}{12}\right) && \text{use property (c)}\\
\implies \frac{\log(2)}{12\log\left(1+\frac{.05}{12}\right)} &= t && \text{divide}\\
\implies t &\approx 13.89180473.
\end{aligned}
$$

Thus, it would take about 13.9 years for your money to double.

4.7 Exercises

In **Exercises 1-10**, use a calculator to evaluate the function at the given value p. Round your answer to the nearest hundredth.

1. $f(x) = \log_4(x)$; $p = 57.60$.

2. $f(x) = \log_4(x)$; $p = 11.22$.

3. $f(x) = \log_7(x)$; $p = 2.98$.

4. $f(x) = \log_3(x)$; $p = 2.27$.

5. $f(x) = \log_6(x)$; $p = 2.56$.

6. $f(x) = \log_8(x)$; $p = 289.27$.

7. $f(x) = \log_8(x)$; $p = 302.67$.

8. $f(x) = \log_5(x)$; $p = 15.70$.

9. $f(x) = \log_8(x)$; $p = 46.13$.

10. $f(x) = \log_4(x)$; $p = 15.59$.

In **Exercises 11-18**, perform each of the following tasks.

a) Approximate the solution of the given equation using your graphing calculator. Load each side of the equation into the `Y=` menu of your calculator. Adjust the `WINDOW` parameters so that the point of intersection of the graphs is visible in the viewing window. Use the `intersect` utility in the `CALC` menu of your calculator to determine the x-coordinate of the point of intersection. Then make an accurate copy of the image in your viewing window on your homework paper.

b) Solve the given equation algebraically, and round your answer to the nearest hundredth.

11. $20 = 3(1.2)^x$

12. $15 = 2(1.8)^x$

13. $14 = 1.4^{5x}$

14. $16 = 1.8^{4x}$

15. $-4 = 0.2^x - 9$

16. $12 = 2.9^x + 2$

17. $13 = 0.1^{x+1}$

18. $19 = 1.2^{x-6}$

In **Exercises 19-34**, solve the given equation algebraically, and round your answer to the nearest hundredth.

19. $20 = e^{x-3}$

20. $-4 = e^x - 9$

21. $23 = 0.9^x + 9$

22. $10 = e^x + 7$

23. $19 = e^x + 5$

24. $4 = 7(2.3)^x$

25. $18 = e^{x+4}$

26. $15 = e^{x+6}$

27. $8 = 2.7^{3x}$

28. $7 = e^{x+1}$

29. $7 = 1.1^{8x}$

30. $6 = 0.2^{x-8}$

31. $-7 = 1.3^{x} - 9$

32. $11 = 3(0.7)^{x}$

33. $23 = e^{x} + 9$

34. $20 = 3.2^{x+1}$

35. Suppose that you invest $17,000 at 6% interest compounded daily. How many years will it take for your investment to double? Round your answer to the nearest hundredth.

36. Suppose that you invest $6,000 at 9% interest compounded continuously. How many years will it take for your investment to double? Round your answer to the nearest hundredth.

37. Suppose that you invest $16,000 at 6% interest compounded daily. How many years will it take for your investment to reach $26,000? Round your answer to the nearest hundredth.

38. Suppose that you invest $15,000 at 5% interest compounded monthly. How many years will it take for your investment to double? Round your answer to the nearest hundredth.

39. Suppose that you invest $18,000 at 3% interest compounded monthly. How many years will it take for your investment to double? Round your answer to the nearest hundredth.

40. Suppose that you invest $7,000 at 5% interest compounded daily. How many years will it take for your investment to reach $13,000? Round your answer to the nearest hundredth.

41. Suppose that you invest $16,000 at 9% interest compounded continuously. How many years will it take for your investment to double? Round your answer to the nearest hundredth.

42. Suppose that you invest $16,000 at 2% interest compounded continuously. How many years will it take for your investment to reach $25,000? Round your answer to the nearest hundredth.

43. Suppose that you invest $2,000 at 5% interest compounded continuously. How many years will it take for your investment to reach $10,000? Round your answer to the nearest hundredth.

44. Suppose that you invest $4,000 at 6% interest compounded continuously. How many years will it take for your investment to reach $10,000? Round your answer to the nearest hundredth.

45. Suppose that you invest $4,000 at 3% interest compounded daily. How many years will it take for your investment to reach $14,000? Round your answer to the nearest hundredth.

46. Suppose that you invest $13,000 at 2% interest compounded monthly. How many years will it take for your investment to reach $20,000? Round your answer to the nearest hundredth.

47. Suppose that you invest $20,000 at 7% interest compounded continuously. How many years will it take for your investment to reach $30,000? Round your answer to the nearest hundredth.

48. Suppose that you invest \$16,000 at 4% interest compounded continuously. How many years will it take for your investment to double? Round your answer to the nearest hundredth.

49. Suppose that you invest \$8,000 at 8% interest compounded continuously. How many years will it take for your investment to double? Round your answer to the nearest hundredth.

50. Suppose that you invest \$3,000 at 3% interest compounded daily. How many years will it take for your investment to double? Round your answer to the nearest hundredth.

4.7 *Answers*

1. 2.92

3. 0.56

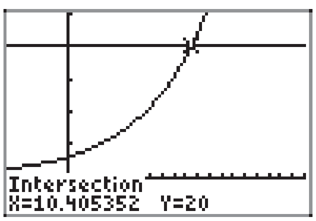
5. 0.52

7. 2.75

9. 1.84

11.

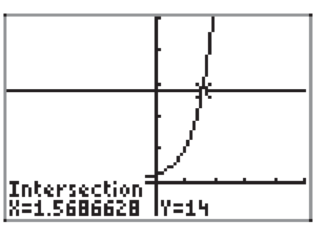
a)



b) 10.41

13.

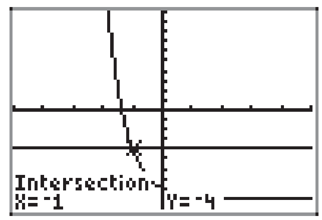
a)



b) 1.57

15.

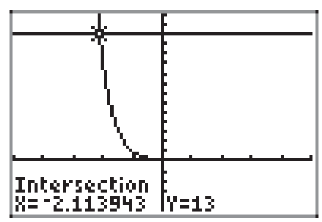
a)



b) -1.00

17.

a)



b) -2.11

19. 6.00

21. -25.05

23. 2.64

25. -1.11

27. 0.70

29. 2.55

31. 2.64

33. 2.64

35. 11.55 years

37. 8.09 years

39. 23.13 years

41. 7.70 years

43. 32.19 years

45. 41.76 years

47. 5.79 years

49. 8.66 years

4.8 Exponential Growth and Decay

Exponential Growth Models

Recalling the investigations in Section 8.3, we started by developing a formula for discrete compound interest. This led to another formula for continuous compound interest,

$$P(t) = P_0 e^{rt}, \tag{1}$$

where P_0 is the initial amount (principal) and r is the annual interest rate in decimal form. If money in a bank account grows at an annual rate r (via payment of interest), and if the growth is continually added in to the account (i.e., interest is continuously compounded), then the balance in the account at time t years is $P(t)$, as given by formula (**1**).

But we can use the exact same analysis for quantities other than money. If $P(t)$ represents the amount of some quantity at time t years, and if $P(t)$ grows at an annual rate r with the growth continually added in, then we can conclude in the same manner that $P(t)$ must have the form

$$P(t) = P_0 e^{rt}, \tag{2}$$

where P_0 is the initial amount at time $t = 0$, namely $P(0)$.

A classic example is *uninhibited population growth*. If a population $P(t)$ of a certain species is placed in a good environment, with plenty of nutrients and space to grow, then it will grow according to formula (**2**). For example, the size of a bacterial culture in a petri dish will follow this formula very closely if it is provided with optimal living conditions. Many other species of animals and plants will also exhibit this behavior if placed in an environment in which they have no predators. For example, when the British imported rabbits into Australia in the late 18th century for hunting, the rabbit population exploded because conditions were good for living and reproducing, and there were no natural predators of the rabbits.

Exponential Growth

If a function $P(t)$ grows continually at a rate $r > 0$, then $P(t)$ has the form

$$P(t) = P_0 e^{rt}, \tag{3}$$

where P_0 is the initial amount $P(0)$. In this case, the quantity $P(t)$ is said to exhibit *exponential growth*, and r is the *growth rate*.

Remarks 4.

1. If a physical quantity (such as population) grows according to formula (**3**), we say that the quantity is *modeled* by the exponential growth function $P(t)$.
2. Some may argue that population growth of rabbits, or even bacteria, is not really continuous. After all, rabbits are born one at a time, so the population actually grows in discrete chunks. This is certainly true, but if the population is large, then the growth will appear to be continuous. For example, consider the world population of humans. There are so many people in the world that there are many new births and deaths each second. Thus, the time difference between each 1 unit change in the population is just a tiny fraction of a second, and consequently the discrete growth will act virtually the same as continuous growth. (This is analogous to the almost identical results for continuous compounding and discrete daily compounding that we found in Section 8.3; compounding each second or millisecond would be even closer.)
3. Likewise, using the continuous exponential growth formula (**3**) to model discrete quantities will sometimes result in fractional answers. In this case, the results will need to be rounded off in order to make sense. For example, an answer of 224.57 rabbits is not actually possible, so the answer should be rounded to 225.
4. In formula (**3**), if time is measured in years (as we have done so far in this chapter), then r is the annual growth rate. However, time can instead be measured in any convenient units. The same formula applies, except that the growth rate r is given in terms of the particular time units used. For example, if time t is measured in hours, then r is the hourly growth rate.

In Section 8.2, we showed that a function of the form b^t with $b > 1$ is an exponential growth function. Likewise, if $A > 0$, then the more general exponential function Ab^t also exhibits exponential growth, since the graph of Ab^t is just a vertical scaling of the graph of b^t. However, the exponential growth function in formula (**3**) appears to be different. We will show below that the function P_0e^{rt} can in fact be written in the form Ab^t with $b > 1$.

Let's first look at a specific example. Suppose $P(t) = 4e^{0.8t}$. Using the Laws of Exponents, we can rewrite $P(t)$ as

$$P(t) = 4e^{0.8t} = 4(e^{0.8})^t. \tag{5}$$

Since $e^{0.8} \approx 2.22554$, it follows that

$$P(t) \approx 4(2.22554)^t.$$

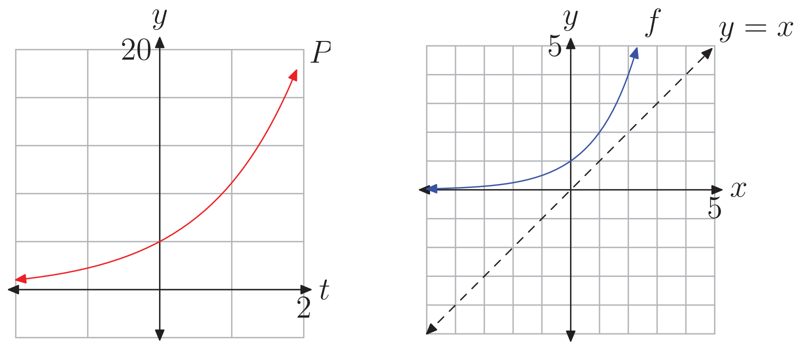
Because the base ≈ 2.22554 is larger than 1, this shows that $P(t)$ is an exponential growth function, as seen in **Figure 1**(a)).

Now suppose that $P(t)$ is any function of the form P_0e^{rt} with $r > 0$. As in (**5**) above, we can use the Laws of Exponents to rewrite $P(t)$ as

$$P(t) = P_0e^{rt} = P_0(e^r)^t = P_0b^t \quad \text{with} \quad b = e^r.$$

To prove that $b > 1$, consider the graph of $y = e^x$ shown in **Figure 1**(b). Recall that $e \approx 2.718$, so $e > 1$, and therefore $y = e^x$ is itself an exponential growth curve. Also, the y-intercept is $(0, 1)$ since $e^0 = 1$. It follows that $b = e^r > 1$ since $r > 0$ (see **Figure 1**(b)).

Therefore, functions of the form $P(t) = P_0 e^{rt}$ with $r > 0$ are exponential growth functions.



(a) $P(t) = 4e^{0.8t}$ (b) $b = e^r > 1$ since $r > 0$

Figure 1.

Applications of Exponential Growth

We will now examine the role of exponential growth functions in some real-world applications. In the following examples, assume that the population is modeled by an exponential growth function as in formula (**3**).

I **Example 6.** *Suppose that the population of a certain country grows at an annual rate of 2%. If the current population is 3 million, what will the population be in 10 years?*

This is a future value problem. If we measure population in millions and time in years, then $P(t) = P_0 e^{rt}$ with $P_0 = 3$ and $r = 0.02$. Inserting these particular values into formula (**3**), we obtain

$$P(t) = 3e^{0.02t}.$$

The population in 10 years is $P(10) = 3e^{(0.02)(10)} \approx 3.664208$ million.

I **Example 7.** *In the same country as in* **Example 6**, *how long will it take the population to reach 5 million?*

As before,

$$P(t) = 3e^{0.02t}.$$

Now we want to know when the future value $P(t)$ of the population at some time t will equal 5 million. Therefore, we need to solve the equation $P(t) = 5$ for time t, which leads to the exponential equation

$$5 = 3e^{0.02t}.$$

Using the procedure for solving exponential equations that was presented in Section 8.6,

$$\begin{aligned}
& 5 = 3e^{0.02t} & \\
\Longrightarrow\ & \frac{5}{3} = e^{0.02t} & \text{isolate the exponential} \\
\Longrightarrow\ & \ln\left(\frac{5}{3}\right) = \ln(e^{0.02t}) & \text{apply the natural log function} \\
\Longrightarrow\ & \ln\left(\frac{5}{3}\right) = 0.02t & \text{since } \ln(e^x) = x \\
\Longrightarrow\ & \frac{\ln\left(\frac{5}{3}\right)}{0.02} = t & \text{division} \\
\Longrightarrow\ & t \approx 25.54128. &
\end{aligned}$$

Thus, it would take about 25.54 years for the population to reach 5 million.

The population of bacteria is typically measured by weight, as in the next two examples.

I **Example 8.** *Suppose that a size of a bacterial culture is given by the function*

$$P(t) = 100e^{0.15t},$$

where the size $P(t)$ is measured in grams and time t is measured in hours. How long will it take for the culture to double in size?

The initial size is $P_0 = 100$ grams, so we want to know when the future value $P(t)$ at some time t will equal 200. Therefore, we need to solve the equation $P(t) = 200$ for time t, which leads to the exponential equation

$$200 = 100e^{0.15t}.$$

Using the same procedure as in the last example,

$$\begin{aligned}
& 200 = 100e^{0.15t} & \\
\Longrightarrow\ & 2 = e^{0.15t} & \text{isolate the exponential} \\
\Longrightarrow\ & \ln(2) = \ln(e^{0.15t}) & \text{apply the natural log function} \\
\Longrightarrow\ & \ln(2) = 0.15t & \text{since } \ln(e^x) = x \\
\Longrightarrow\ & \frac{\ln(2)}{0.15} = t & \text{division} \\
\Longrightarrow\ & t \approx 4.620981. &
\end{aligned}$$

Thus, it would take about 4.62 hours for the size to double.

The last example deserves an additional comment. Suppose that we had started with 1000 grams instead of 100. Then to double in size would require a future value of 2000 grams. Therefore, in this case, we would have to solve the equation

$$2000 = 1000e^{0.15t}.$$

But the first step is to isolate the exponential by dividing both sides by 1000 to get

$$2 = e^{0.15t},$$

and this is the same as the second line of the solution in the last example, so the answer will be the same. Likewise, repeating this argument for any initial amount will lead to the same second line, and therefore the same answer. Thus, the *doubling time* depends only on r, not on the initial amount P_0.

Exponential Decay Models

We've observed that if a quantity increases continually at a rate r, then it is modeled by a function of the form $P(t) = P_0 e^{rt}$. But what if a quantity *decreases* instead? Although we won't present the details here, the analysis can be carried out in the same way as the derivation of the continuous compounding formula in Section 8.3. The only difference is that the growth rate r in the formulas must be replaced by $-r$ since the quantity is decreasing. The conclusion is that the quantity is modeled by a function of the form $P(t) = P_0 e^{-rt}$ instead of $P_0 e^{rt}$.

Exponential Decay

If a function $P(t)$ decreases continually at a rate $r > 0$, then $P(t)$ has the form

$$P(t) = P_0 e^{-rt}, \tag{9}$$

where P_0 is the initial amount $P(0)$. In this case, the quantity $P(t)$ is said to exhibit *exponential decay*, and r is the *decay rate*.

In Section 8.2, we showed that a function of the form b^t with $b < 1$ is an exponential decay function. Likewise, if $A > 0$, then the more general exponential function Ab^t also exhibits exponential decay, since the graph of Ab^t is just a vertical scaling of the graph of b^t. However, the exponential decay function in formula (**9**) appears to be different. We will show below that the function $P_0 e^{-rt}$ can in fact be written in the form Ab^t with $b < 1$.

Let's first look at a specific example. Suppose $P(t) = 4e^{-0.8t}$. Using the Laws of Exponents, we can rewrite $P(t)$ as

$$P(t) = 4e^{-0.8t} = 4(e^{-0.8})^t. \tag{10}$$

Since $e^{-0.8} \approx 0.44933$, it follows that

$$P(t) \approx 4(0.44933)^t.$$

Because the base ≈ 0.44933 is less than 1, this shows that $P(t)$ is an exponential decay function, as seen in **Figure 2**(a)).

Now suppose that $P(t)$ is any function of the form P_0e^{-rt} with $r > 0$. As in (**10**) above, we can use the Laws of Exponents to rewrite $P(t)$ as

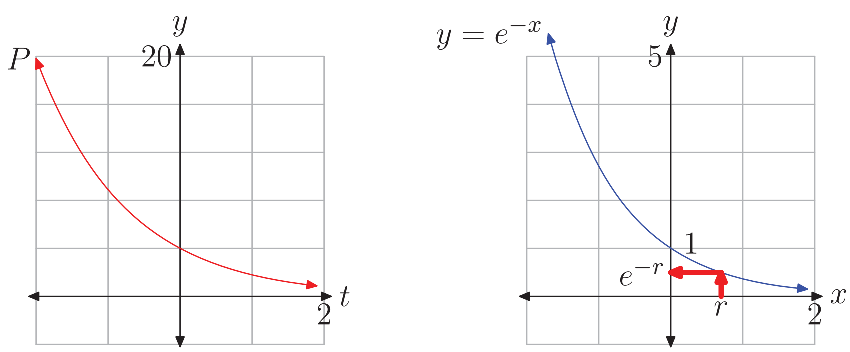
$$P(t) = P_0e^{-rt} = P_0(e^{-r})^t = P_0b^t \quad \text{with} \quad b = e^{-r}.$$

To prove that $b < 1$, consider the graph of $y = e^{-x}$ shown in **Figure 2**(b). Now

$$e^{-x} = (e^{-1})^x = \left(\frac{1}{e}\right)^x$$

and $1/e \approx 0.36788 < 1$, so $y = e^{-x}$ is itself an exponential decay curve. (Alternatively, you can observe that the graph of $y = e^{-x}$ is the reflection of the graph of $y = e^x$ across the y-axis.) Also, the y-intercept is $(0, 1)$ since $e^{-0} = 1$. It follows that $b = e^{-r} < 1$ since $r > 0$ (see **Figure 2**(b)).

Therefore, functions of the form $P(t) = P_0e^{-rt}$ with $r > 0$ are exponential decay functions.



(a) $P(t) = 4e^{-0.8t}$ (b) $b = e^{-r} < 1$ since $r > 0$

Figure 2.

Applications of Exponential Decay

The main example of exponential decay is *radioactive decay*. Radioactive elements and isotopes spontaneously emit subatomic particles, and this process gradually changes the substance into a different isotope. For example, the radioactive isotope Uranium-238 eventually decays into the stable isotope Lead-206. This is a random process for individual atoms, but overall the mass of the substance decreases according to the exponential decay formula (**9**).

I **Example 11.** *Suppose that a certain radioactive element has an annual decay rate of 10%. Starting with a 200 gram sample of the element, how many grams will be left in 3 years?*

This is a future value problem. If we measuring size in grams and time in years, then $P(t) = P_0 e^{-rt}$ with $P_0 = 200$ and $r = 0.10$. Inserting these particular values into formula (**9**), we obtain

$$P(t) = 200e^{-0.10t}.$$

The amount in 3 years is $P(3) = 200e^{-(0.10)(3)} \approx 148.1636$ grams.

I **Example 12.** *Using the same element as in* **Example 11**, *if a particular sample of the element decays to 50 grams after 5 years, how big was the original sample?*

This is a present value problem, where the unknown is the initial amount P_0. As before, $r = 0.10$, so

$$P(t) = P_0 e^{-0.10t}.$$

Since $P(5) = 50$, we have the equation

$$50 = P(5) = P_0 e^{-(0.10)(5)}.$$

This equation can be solved by division:

$$\frac{50}{e^{-(0.10)(5)}} = P_0$$

Finish by calculating the value of the left side to get $P_0 \approx 82.43606$ grams.

I **Example 13.** *Suppose that a certain radioactive isotope has an annual decay rate of 5%. How many years will it take for a 100 gram sample to decay to 40 grams?*

Use $P(t) = P_0 e^{-rt}$ with $P_0 = 100$ and $r = 0.05$, so

$$P(t) = 100e^{-0.05t}.$$

Now we want to know when the future value $P(t)$ of the size of the sample at some time t will equal 40. Therefore, we need to solve the equation $P(t) = 40$ for time t, which leads to the exponential equation

$$40 = 100e^{-0.05t}.$$

Using the procedure for solving exponential equations that was presented in Section 8.6,

$$\begin{aligned}
& 40 = 100e^{-0.05t} & \\
\Longrightarrow\ & 0.4 = e^{-0.05t} & \text{isolate the exponential} \\
\Longrightarrow\ & \ln(0.4) = \ln(e^{-0.05t}) & \text{apply the natural log function} \\
\Longrightarrow\ & \ln(0.4) = -0.05t & \text{since } \ln(e^x) = x \\
\Longrightarrow\ & \frac{\ln(0.4)}{-0.05} = t & \text{division} \\
\Longrightarrow\ & t \approx 18.32581. &
\end{aligned}$$

Thus, it would take approximately 18.33 years for the sample to decay to 40 grams.

We saw earlier that exponential growth processes have a fixed doubling time. Similarly, exponential decay processes have a fixed *half-life*, the time in which one-half the original amount decays.

I **Example 14.** *Using the same element as in* **Example 13**, *what is the half-life of the element?*

As before, $r = 0.05$, so

$$P(t) = P_0 e^{-0.05t}.$$

The initial size is P_0 grams, so we want to know when the future value $P(t)$ at some time t will equal one-half the initial amount, $P_0/2$. Therefore, we need to solve the equation $P(t) = P_0/2$ for time t, which leads to the exponential equation

$$\frac{P_0}{2} = P_0 e^{-0.05t}.$$

Using the same procedure as in the last example,

$$\begin{aligned}
& \frac{P_0}{2} = P_0 e^{-0.05t} & \\
\Longrightarrow\ & \frac{1}{2} = e^{-0.05t} & \text{isolate the exponential} \\
\Longrightarrow\ & \ln\left(\frac{1}{2}\right) = \ln(e^{-0.05t}) & \text{apply the natural log function} \\
\Longrightarrow\ & \ln\left(\frac{1}{2}\right) = -0.05t & \text{since } \ln(e^x) = x \\
\Longrightarrow\ & \frac{\ln\left(\frac{1}{2}\right)}{-0.05} = t & \text{division} \\
\Longrightarrow\ & t \approx 13.86294. &
\end{aligned}$$

Thus, the half-life is approximately 13.86 years.

The process of radioactive decay also forms the basis of the carbon-14 dating technique. The Earth's atmosphere contains a tiny amount of the radioactive isotope carbon-14, and therefore plants and animals also contain some carbon-14 due to their interaction with the atmosphere. However, this interaction ends when a plant or animal dies, so the carbon-14 begins to decay (the decay rate is 0.012%). By comparing the amount of carbon-14 in a bone, for example, with the normal amount in a living animal, scientists can compute the age of the bone.

I **Example 15.** *Suppose that only 1.5% of the normal amount of carbon-14 remains in a fragment of bone. How old is the bone?*

Use $P(t) = P_0 e^{-rt}$ with $r = 0.00012$, so

$$P(t) = P_0 e^{-0.00012t}.$$

The initial size is P_0 grams, so we want to know when the future value $P(t)$ at some time t will equal 1.5% of the initial amount, $0.015P_0$. Therefore, we need to solve the equation $P(t) = 0.015P_0$ for time t, which leads to the exponential equation

$$0.015P_0 = P_0 e^{-0.00012t}.$$

Using the same procedure as in **Example 14**,

$$\begin{aligned}
& 0.015P_0 = P_0 e^{-0.00012t} && \\
\Longrightarrow\ & 0.015 = e^{-0.00012t} && \text{isolate the exponential} \\
\Longrightarrow\ & \ln(0.015) = \ln(e^{-0.00012t}) && \text{apply the natural log function} \\
\Longrightarrow\ & \ln(0.015) = -0.00012t && \text{since } \ln(e^x) = x \\
\Longrightarrow\ & \frac{\ln(0.015)}{-0.00012} = t && \text{division} \\
\Longrightarrow\ & t \approx 34998. &&
\end{aligned}$$

Thus, the bone is approximately 34998 years old.

While the carbon-14 technique only works on plants and animals, there are other similar dating techniques, using other radioactive isotopes, that are used to date rocks and other inorganic matter.

4.8 *Exercises*

1. Suppose that the population of a certain town grows at an annual rate of 6%. If the population is currently 5,000, what will it be in 7 years? Round your answer to the nearest integer.

2. Suppose that the population of a certain town grows at an annual rate of 5%. If the population is currently 2,000, how many years will it take for it to double? Round your answer to the nearest hundredth.

3. Suppose that a certain radioactive isotope has an annual decay rate of 7.2%. How many years will it take for a 227 gram sample to decay to 93 grams? Round your answer to the nearest hundredth.

4. Suppose that a certain radioactive isotope has an annual decay rate of 6.8%. How many years will it take for a 399 gram sample to decay to 157 grams? Round your answer to the nearest hundredth.

5. Suppose that the population of a certain town grows at an annual rate of 8%. If the population is currently 4,000, how many years will it take for it to double? Round your answer to the nearest hundredth.

6. Suppose that a certain radioactive isotope has an annual decay rate of 19.2%. Starting with a 443 gram sample, how many grams will be left after 9 years? Round your answer to the nearest hundredth.

7. Suppose that a certain radioactive isotope has an annual decay rate of 17.4%. What is the half-life (in years) of the isotope? Round your answer to the nearest hundredth.

8. Suppose that the population of a certain town grows at an annual rate of 7%. If the population is currently 8,000, how many years will it take for it to reach 18,000? Round your answer to the nearest hundredth.

9. Suppose that a certain radioactive isotope has an annual decay rate of 17.3%. Starting with a 214 gram sample, how many grams will be left after 5 years? Round your answer to the nearest hundredth.

10. Suppose that the population of a certain town grows at an annual rate of 7%. If the population grows to 2,000 in 7 years, what was the original population? Round your answer to the nearest integer.

11. Suppose that the population of a certain town grows at an annual rate of 3%. If the population is currently 3,000, how many years will it take for it to double? Round your answer to the nearest hundredth.

12. Suppose that a certain radioactive isotope has an annual decay rate of 12.5%. Starting with a 127 gram sample, how many grams will be left after 6 years? Round your answer to the nearest hundredth.

13. Suppose that a certain radioactive isotope has an annual decay rate of 13.1%.

Starting with a 353 gram sample, how many grams will be left after 7 years? Round your answer to the nearest hundredth.

14. Suppose that the population of a certain town grows at an annual rate of 2%. If the population grows to $9,000$ in 4 years, what was the original population? Round your answer to the nearest integer.

15. Suppose that the population of a certain town grows at an annual rate of 2%. If the population is currently 7,000, how many years will it take for it to double? Round your answer to the nearest hundredth.

16. Suppose that a certain radioactive isotope has an annual decay rate of 5.3%. How many years will it take for a 217 gram sample to decay to 84 grams? Round your answer to the nearest hundredth.

17. Suppose that a certain radioactive isotope has an annual decay rate of 18.7%. How many years will it take for a 324 gram sample to decay to 163 grams? Round your answer to the nearest hundredth.

18. Suppose that the population of a certain town grows at an annual rate of 8%. If the population is currently 8,000, how many years will it take for it to reach 18,000? Round your answer to the nearest hundredth.

19. Suppose that a certain radioactive isotope has an annual decay rate of 2.3%. If a particular sample decays to 25 grams after 8 years, how big (in grams) was the original sample? Round your answer to the nearest hundredth.

20. Suppose that the population of a certain town grows at an annual rate of 4%. If the population is currently 7,000, how many years will it take for it to reach 17,000? Round your answer to the nearest hundredth.

21. Suppose that a certain radioactive isotope has an annual decay rate of 9.8%. If a particular sample decays to 11 grams after 6 years, how big (in grams) was the original sample? Round your answer to the nearest hundredth.

22. Suppose that the population of a certain town grows at an annual rate of 5%. If the population grows to $6,000$ in 3 years, what was the original population? Round your answer to the nearest integer.

23. Suppose that the population of a certain town grows at an annual rate of 8%. If the population is currently $6,000$, what will it be in 5 years? Round your answer to the nearest integer.

24. Suppose that a certain radioactive isotope has an annual decay rate of 15.8%. What is the half-life (in years) of the isotope? Round your answer to the nearest hundredth.

25. Suppose that the population of a certain town grows at an annual rate of 9%. If the population grows to $7,000$ in 5 years, what was the original population? Round your answer to the nearest integer.

26. Suppose that a certain radioactive isotope has an annual decay rate of 18.6%. If a particular sample decays to 41 grams after 3 years, how big (in grams) was the original sample? Round your answer to the nearest hundredth.

27. Suppose that a certain radioactive isotope has an annual decay rate of 5.2%. What is the half-life (in years) of the isotope? Round your answer to the nearest hundredth.

28. Suppose that a certain radioactive isotope has an annual decay rate of 6.5%. What is the half-life (in years) of the isotope? Round your answer to the nearest hundredth.

29. Suppose that the population of a certain town grows at an annual rate of 8%. If the population is currently 2,000, how many years will it take for it to reach 7,000? Round your answer to the nearest hundredth.

30. Suppose that a certain radioactive isotope has an annual decay rate of 3.7%. If a particular sample decays to 47 grams after 8 years, how big (in grams) was the original sample? Round your answer to the nearest hundredth.

31. Suppose that the population of a certain town grows at an annual rate of 6%. If the population is currently $7,000$, what will it be in 7 years? Round your answer to the nearest integer.

32. Suppose that the population of a certain town grows at an annual rate of 4%. If the population is currently $1,000$, what will it be in 3 years? Round your answer to the nearest integer.

In **Exercises 33-40**, use the fact that the decay rate of carbon-14 is 0.012%. Round your answer to the nearest year.

33. Suppose that only 8.6% of the normal amount of carbon-14 remains in a fragment of bone. How old is the bone?

34. Suppose that only 5.2% of the normal amount of carbon-14 remains in a fragment of bone. How old is the bone?

35. Suppose that 90.1% of the normal amount of carbon-14 remains in a piece of wood. How old is the wood?

36. Suppose that 83.6% of the normal amount of carbon-14 remains in a piece of cloth. How old is the cloth?

37. Suppose that only 6.2% of the normal amount of carbon-14 remains in a fragment of bone. How old is the bone?

38. Suppose that only 1.3% of the normal amount of carbon-14 remains in a fragment of bone. How old is the bone?

39. Suppose that 96.7% of the normal amount of carbon-14 remains in a piece of cloth. How old is the cloth?

40. Suppose that 84.9% of the normal amount of carbon-14 remains in a piece of wood. How old is the wood?

4.8 *Answers*

1. 7610 people

3. 12.39 yrs

5. 8.66 yrs

7. 3.98 yrs

9. 90.11g

11. 23.10 yrs

13. 141.10g

15. 34.66 yrs

17. 3.67 yrs

19. 30.05g

21. 19.80g

23. $8,951$ people

25. $4,463$ people

27. 13.33 yrs

29. 15.66 yrs

31. $10,654$ people

33. 20445 years

35. 869 years

37. 23172 years

39. 280 years

5 Radical Functions

In this chapter, we will study *radical functions* — in other words, functions that involve square, cubic, and other roots of algebraic expressions (for example, $\sqrt{x}$ or $\sqrt[3]{x+2}$). There are a number of subtleties and tricks to these functions, and it is important to learn how to manipulate them.

Radical functions are closely related to *power functions* (for example, x^2 or $(2-x)^5$). In fact, the graph of $\sqrt{x}$ is exactly what you would see if you reflected the graph of x^2 across the line $y = x$ and erased everything below the x-axis! It turns out that $\sqrt{x}$ is so closely related to x^2 that we say that those functions are *inverses* of each other; whatever one does, the other undoes.

Radical functions have many interesting applications, are studied extensively in many mathematics courses, and are used often in science and engineering. If you have ever wanted to calculate the shortest distance between two places, or predict how long a stairway is based upon the height it reaches, radical functions can help you with these calculations.

5.1 The Square Root Function

In this section we turn our attention to the square root function, the function defined by the equation

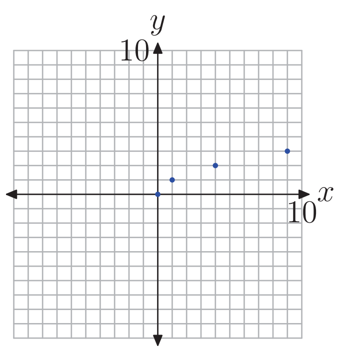
$$f(x) = \sqrt{x}. \tag{1}$$

We begin the section by drawing the graph of the function, then we address the domain and range. After that, we'll investigate a number of different transformations of the function.

The Graph of the Square Root Function

Let's create a table of points that satisfy the equation of the function, then plot the points from the table on a Cartesian coordinate system on graph paper. We'll continue creating and plotting points until we are convinced of the eventual shape of the graph.

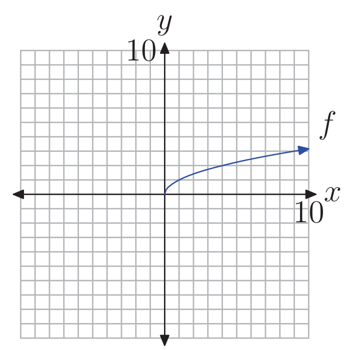
We know we cannot take the square root of a negative number. Therefore, we don't want to put any negative x-values in our table. To further simplify our computations, let's use numbers whose square root is easily calculated. This brings to mind perfect squares such as 0, 1, 4, 9, and so on. We've placed these numbers as x-values in the table in **Figure 1**(b), then calculated the square root of each. In **Figure 1**(a), you see each of the points from the table plotted as a solid dot. If we continue to add points to the table, plot them, the graph will eventually fill in and take the shape of the solid curve shown in **Figure 1**(c).



(a)

x	$f(x) = \sqrt{x}$
0	0
1	1
4	2
9	3

(b)

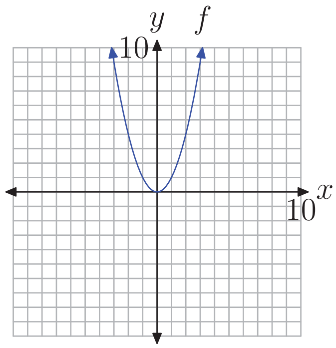


(c)

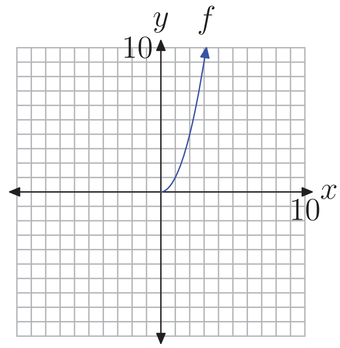
Figure 1. Creating the graph of $f(x) = \sqrt{x}$.

The point plotting approach used to draw the graph of $f(x) = \sqrt{x}$ in **Figure 1** is a tested and familiar procedure. However, a more sophisticated approach involves the theory of inverses developed in the previous chapter.

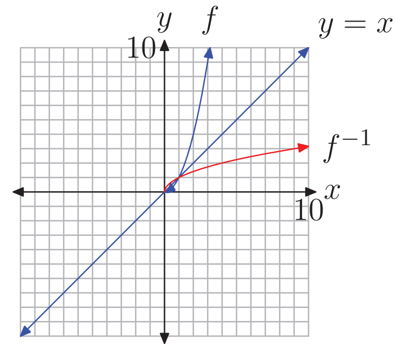
In a sense, taking the square root is the "inverse" of squaring. Well, not quite, as the squaring function $f(x) = x^2$ in **Figure 2**(a) fails the horizontal line test and is not one-to-one. However, if we limit the domain of the squaring function, then the graph of $f(x) = x^2$ in **Figure 2**(b), where $x \geq 0$, does pass the horizontal line test and is one-to-one. Therefore, the graph of $f(x) = x^2$, $x \geq 0$, has an inverse, and the graph of its inverse is found by reflecting the graph of $f(x) = x^2$, $x \geq 0$, across the line $y = x$ (see **Figure 2**(c)).



(a) $f(x) = x^2$.



(b) $f(x) = x^2$, $x \geq 0$.



(c) Reflecting the graph in (b) across the line $y = x$ produces the graph of $f^{-1}(x) = \sqrt{x}$.

Figure 2. Sketching the inverse of $f(x) = x^2$, $x \geq 0$.

To find the equation of the inverse, recall that the procedure requires that we switch the roles of x and y, then solve the resulting equation for y. Thus, first write $f(x) = x^2$, $x \geq 0$, in the form

$$y = x^2, \quad x \geq 0.$$

Next, switch x and y.

$$x = y^2, \quad y \geq 0 \tag{2}$$

When we solve this last equation for y, we get two solutions,

$$y = \pm\sqrt{x}. \tag{3}$$

However, in **equation (2)**, note that y must be greater than or equal to zero. Hence, we must choose the nonnegative answer in **equation (3)**, so the inverse of $f(x) = x^2$, $x \geq 0$, has equation

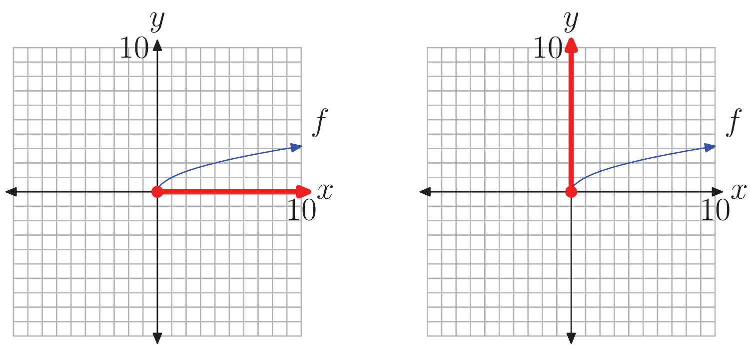
$$f^{-1}(x) = \sqrt{x}.$$

This is the equation of the reflection of the graph of $f(x) = x^2$, $x \geq 0$, that is pictured in **Figure 2**(c). Note the exact agreement with the graph of the square root function in **Figure 1**(c).

The sequence of graphs in **Figure 2** also help us identify the domain and range of the square root function.

- In **Figure 2**(a), the parabola opens outward indefinitely, both left and right. Consequently, the domain is $D_f = (-\infty, \infty)$, or all real numbers. Also, the graph has vertex at the origin and opens upward indefinitely, so the range is $R_f = [0, \infty)$.
- In **Figure 2**(b), we restricted the domain. Thus, the graph of $f(x) = x^2$, $x \geq 0$, now has domain $D_f = [0, \infty)$. The range is unchanged and is $R_f = [0, \infty)$.
- In **Figure 2**(c), we've reflected the graph of $f(x) = x^2$, $x \geq 0$, across the line $y = x$ to obtain the graph of $f^{-1}(x) = \sqrt{x}$. Because we've interchanged the role of x and y, the domain of the square root function must equal the range of $f(x) = x^2$, $x \geq 0$. That is, $D_{f^{-1}} = [0, \infty)$. Similarly, the range of the square root function must equal the domain of $f(x) = x^2$, $x \geq 0$. Hence, $R_{f^{-1}} = [0, \infty)$.

Of course, we can also determine the domain and range of the square root function by projecting all points on the graph onto the x- and y-axes, as shown in **Figures 3**(a) and (b), respectively.



(a) Domain $= [0, \infty)$ (b) Range $= [0, \infty)$

Figure 3. Project onto the axes to find the domain and range.

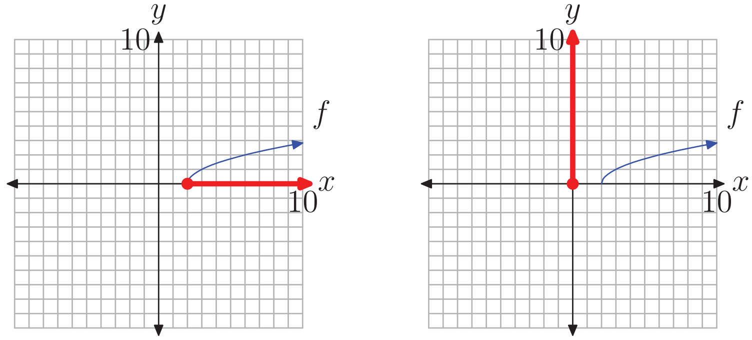
Some might object to the range, asking "How do we know that the graph of the square root function picture in **Figure 3**(b) rises indefinitely?" Again, the answer lies in the sequence of graphs in **Figure 2**. In **Figure 2**(c), note that the graph of $f(x) = x^2$, $x \geq 0$, opens indefinitely to the right as the graph rises to infinity. Hence, after reflecting this graph across the line $y = x$, the resulting graph must rise upward indefinitely as it moves to the right. Thus, the range of the square root function is $[0, \infty)$.

Translations

If we shift the graph of $y = \sqrt{x}$ right and left, or up and down, the domain and/or range are affected.

▶ **Example 4.** *Sketch the graph of $f(x) = \sqrt{x-2}$. Use your graph to determine the domain and range.*

We know that the basic equation $y = \sqrt{x}$ has the graph shown in **Figure 1**(c). If we replace x with $x - 2$, the basic equation $y = \sqrt{x}$ becomes $y = \sqrt{x-2}$. From our previous work with geometric transformations, we know that this will shift the graph two units to the right, as shown in **Figures 4**(a) and (b).



(a) Domain $= [2, \infty)$ (b) Range $= [0, \infty)$

Figure 4. To draw the graph of $f(x) = \sqrt{x-2}$, shift the graph of $y = \sqrt{x}$ two units to the right.

To find the domain, we project each point on the graph of f onto the x-axis, as shown in **Figure 4**(a). Note that all points to the right of or including 2 are shaded on the x-axis. Consequently, the domain of f is

$$\text{Domain} = [2, \infty) = \{x : x \geq 2\}.$$

As there has been no shift in the vertical direction, the range remains the same. To find the range, we project each point on the graph onto the y-axis, as shown in **Figure 4**(b). Note that all points at and above zero are shaded on the y-axis. Thus, the range of f is

$$\text{Range} = [0, \infty) = \{y : y \geq 0\}.$$

We can find the domain of this function algebraically by examining its defining equation $f(x) = \sqrt{x-2}$. We understand that we cannot take the square root of a negative number. Therefore, the expression under the radical must be nonnegative (positive or zero). That is,

$$x - 2 \geq 0.$$

Solving this inequality for x,

$$x \geq 2.$$

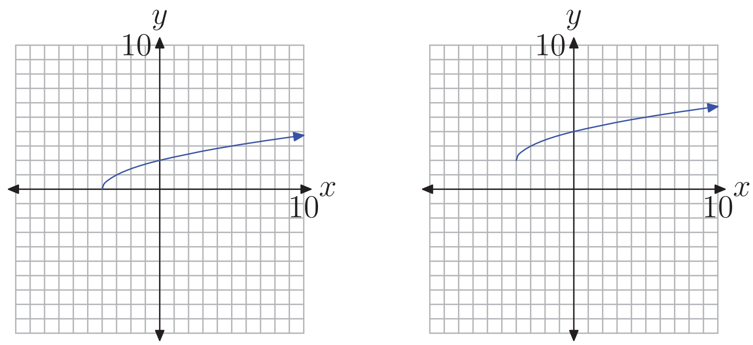
Thus, the domain of f is Domain $= [2, \infty)$, which matches the graphical solution above.

Let's look at another example.

► **Example 5.** *Sketch the graph of* $f(x) = \sqrt{x+4} + 2$. *Use your graph to determine the domain and range of* f.

Again, we know that the basic equation $y = \sqrt{x}$ has the graph shown in **Figure 1**(c). If we replace x with $x + 4$, the basic equation $y = \sqrt{x}$ becomes $y = \sqrt{x+4}$. From our previous work with geometric transformations, we know that this will shift the graph of $y = \sqrt{x}$ four units to the left, as shown in **Figure 5**(a).

If we now add 2 to the equation $y = \sqrt{x+4}$ to produce the equation $y = \sqrt{x+4}+2$, this will shift the graph of $y = \sqrt{x+4}$ two units upward, as shown in **Figure 5**(b).



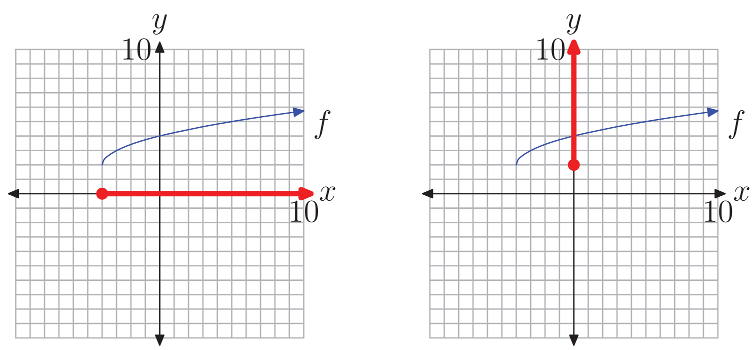
(a) To draw the graph of $y = \sqrt{x+4}$, shift the graph of $y = \sqrt{x}$ four units to the left.

(b) To draw the graph of $y = \sqrt{x+4}+2$, shift the graph of $y = \sqrt{x+4}$ two units upward.

Figure 5. Translating the original equation $y = \sqrt{x}$ to get the graph of $y = \sqrt{x+4}+2$.

To identify the domain of $f(x) = \sqrt{x+4}+2$, we project all points on the graph of f onto the x-axis, as shown in **Figure 6**(a). Note that all points to the right of or including -4 are shaded on the x-axis. Thus, the domain of $f(x) = \sqrt{x+4}+2$ is

$$\text{Domain} = [-4, \infty) = \{x : x \geq -4\}.$$



(a) Shading the domain of f.

(b) Shading the range of f.

Figure 6. Project points of f onto the axes to determine the domain and range.

Similarly, to find the range of f, project all points on the graph of f onto the y-axis, as shown in **Figure 6**(b). Note that all points on the y-axis greater than or including 2 are shaded. Consequently, the range of f is

$$\text{Range} = [2, \infty) = \{y : y \geq 2\}.$$

We can also find the domain of f algebraically by examining the equation $f(x) = \sqrt{x+4}+2$. We cannot take the square root of a negative number, so the expression under the radical must be nonnegative (zero or positive). Consequently,

$$x + 4 \geq 0.$$

Solving this inequality for x,

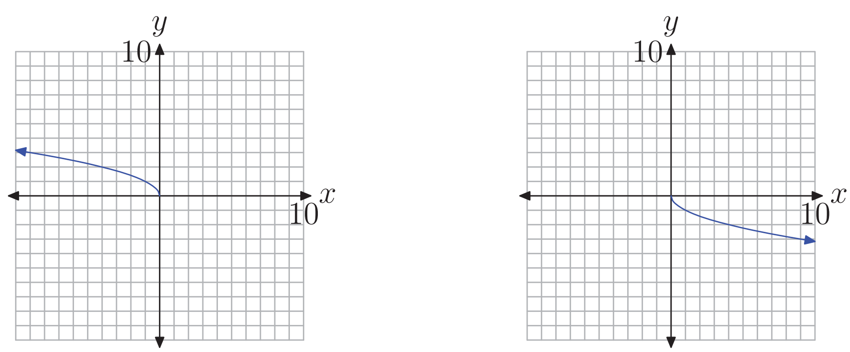
$$x \geq -4.$$

Thus, the domain of f is Domain $= [-4, \infty)$, which matches the graphical solution presented above.

Reflections

If we start with the basic equation $y = \sqrt{x}$, then replace x with $-x$, then the graph of the resulting equation $y = \sqrt{-x}$ is captured by reflecting the graph of $y = \sqrt{x}$ (see **Figure 1**(c)) horizontally across the y-axis. The graph of $y = \sqrt{-x}$ is shown in **Figure 7**(a).

Similarly, the graph of $y = -\sqrt{x}$ would be a vertical reflection of the graph of $y = \sqrt{x}$ across the x-axis, as shown in **Figure 7**(b).



(a) To obtain the graph of $y = \sqrt{-x}$, reflect the graph of $y = \sqrt{x}$ across the y-axis.

(b) To obtain the graph of $y = -\sqrt{x}$, reflect the graph of $y = \sqrt{x}$ across the x-axis.

Figure 7. Reflecting the graph of $y = \sqrt{x}$ across the x- and y-axes.

More often than not, you will be asked to perform a reflection **and** a translation.

▶ **Example 6.** *Sketch the graph of $f(x) = \sqrt{4-x}$. Use the resulting graph to determine the domain and range of f.*

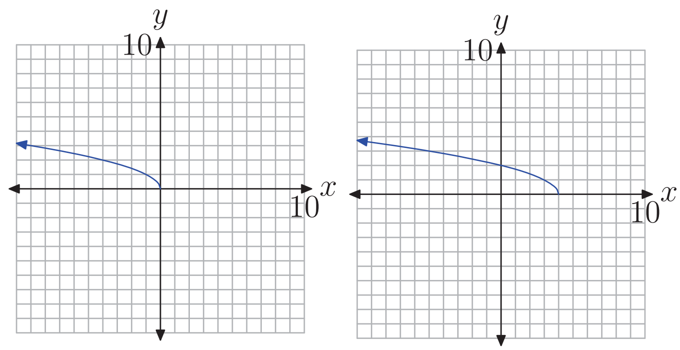
First, rewrite the equation $f(x) = \sqrt{4-x}$ as follows:

$$f(x) = \sqrt{-(x-4)}.$$

Reflections First. It is usually more intuitive to perform reflections before translations.

With this thought in mind, we first sketch the graph of $y = \sqrt{-x}$, which is a reflection of the graph of $y = \sqrt{x}$ across the y-axis. This is shown in **Figure 8**(a).

Now, in $y = \sqrt{-x}$, replace x with $x - 4$ to obtain $y = \sqrt{-(x-4)}$. This shifts the graph of $y = \sqrt{-x}$ four units to the right, as pictured in **Figure 8**(b).



(a) The graph of $y = \sqrt{-x}$.

(b) The graph of $y = \sqrt{-(x-4)}$.

Figure 8. A reflection followed by a translation.

To find the domain of the function $f(x) = \sqrt{-(x-4)}$, or equivalently, $f(x) = \sqrt{4-x}$, project each point on the graph of f onto the x-axis, as shown in **Figure 9**(a). Note that all real numbers less than or equal to 4 are shaded on the x-axis. Hence, the domain of f is

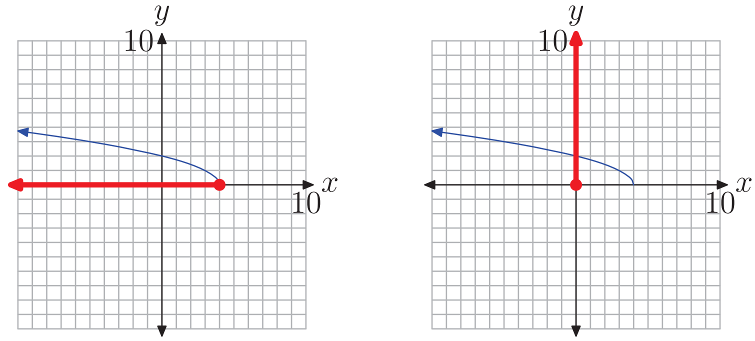
$$\text{Domain} = (-\infty, 4] = \{x : x \leq 4\}.$$

Similarly, to obtain the range of f, project each point on the graph of f onto the y-axis, as shown in **Figure 9**(b). Note that all real numbers greater than or equal to zero are shaded on the y-axis. Hence, the range of f is

$$\text{Range} = [0, \infty) = \{y : y \geq 0\}.$$

We can also find the domain of the function f by examining the equation $f(x) = \sqrt{4-x}$. We cannot take the square root of a negative number, so the expression under the radical must be nonnegative (zero or positive). Consequently,

$$4 - x \geq 0.$$



(a) Project onto the x-axis to determine the domain.

(b) Project onto the y-axis to determine the range.

Figure 9. Determining the domain and range of $f(x) = \sqrt{4-x}$.

Solve this last inequality for x. First subtract 4 from both sides of the inequality, then multiply both sides of the resulting inequality by -1. Of course, multiplying by a negative number reverses the inequality symbol.

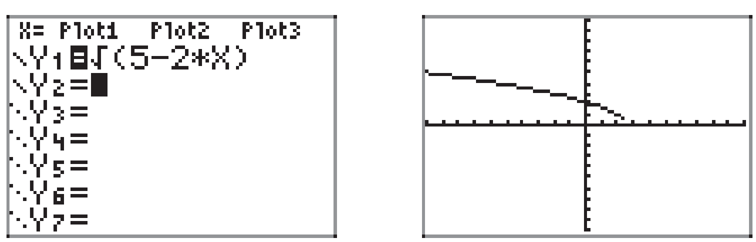
$$\begin{aligned} -x &\geq -4 \\ x &\leq 4 \end{aligned}$$

Thus, the domain of f is $\{x : x \leq 4\}$. In interval notation, Domain $= (-\infty, 4]$. This agree nicely with the graphical result found above.

More often than not, it will take a combination of your graphing calculator and a little algebraic manipulation to determine the domain of a square root function.

▶ **Example 7.** *Sketch the graph of $f(x) = \sqrt{5-2x}$. Use the graph and an algebraic technique to determine the domain of the function.*

Load the function into `Y1` in the `Y=` menu of your calculator, as shown in **Figure 10**(a). Select `6:ZStandard` from the `ZOOM` menu to produce the graph shown in **Figure 10**(b).



(a) (b)

Figure 10. Drawing the graph of $f(x) = \sqrt{5-2x}$ on the graphing calculator.

Look carefully at the graph in **Figure 10**(b) and note that it's difficult to tell if the graph comes all the way down to "touch" the x-axis near $x \approx 2.5$. However, our previous experience with the square root function makes us believe that this is just an artifact of insufficient resolution on the calculator that is preventing the graph from "touching" the x-axis at $x \approx 2.5$.

An algebraic approach will settle the issue. We can determine the domain of f by examining the equation $f(x) = \sqrt{5-2x}$. We cannot take the square root of a negative number, so the expression under the radical must be nonnegative (zero or positive). Consequently,

$$5 - 2x \geq 0.$$

Solve this last inequality for x. First, subtract 5 from both sides of the inequality.

$$-2x \geq -5$$

Next, divide both sides of this last inequality by -2. Remember that we must reverse the inequality the moment we divide by a negative number.

$$\begin{aligned} \frac{-2x}{-2} &\leq \frac{-5}{-2} \\ x &\leq \frac{5}{2} \end{aligned}$$

Thus, the domain of f is $\{x : x \leq 5/2\}$. In interval notation, Domain $= (-\infty, 5/2]$.

Further introspection reveals that this argument also settles the issue of whether or not the graph "touches" the x-axis at $x = 5/2$. If you remain unconvinced, then substitute $x = 5/2$ in $f(x) = \sqrt{5-2x}$ to see

$$f(5/2) = \sqrt{5 - 2(5/2)} = \sqrt{0} = 0.$$

Thus, the graph of f "touches" the x-axis at the point $(5/2, 0)$.

5.1 Exercises

In **Exercises 1-10**, complete each of the following tasks.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis.
ii. Complete the table of points for the given function. Plot each of the points on your coordinate system, then use them to help draw the graph of the given function.
iii. Use different colored pencils to project all points onto the x- and y-axes to determine the domain and range. Use interval notation to describe the domain of the given function.

1. $f(x) = -\sqrt{x}$

x	0	1	4	9
$f(x)$				

2. $f(x) = \sqrt{-x}$

x	0	-1	-4	-9
$f(x)$				

3. $f(x) = \sqrt{x+2}$

x	-2	-1	2	7
$f(x)$				

4. $f(x) = \sqrt{5-x}$

x	-4	1	4	5
$f(x)$				

5. $f(x) = \sqrt{x} + 2$

x	0	1	4	9
$f(x)$				

6. $f(x) = \sqrt{x} - 1$

x	0	1	4	9
$f(x)$				

7. $f(x) = \sqrt{x+3} + 2$

x	-3	-2	1	6
$f(x)$				

8. $f(x) = \sqrt{x-1} + 3$

x	1	2	5	10
$f(x)$				

9. $f(x) = \sqrt{3-x}$

x	-6	-1	2	3
$f(x)$				

10. $f(x) = -\sqrt{x+3}$

x	-3	-2	1	6
$f(x)$				

In **Exercises 11-20**, perform each of the following tasks.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. *Remember to draw all lines with a ruler.*
ii. Use geometric transformations to draw the graph of the given function on your coordinate system without the use of a graphing calculator. *Note: You may* ***check*** *your solution with your calculator, but you should be able to produce the graph without the use of your calculator.*
iii. Use different colored pencils to project the points on the graph of the function onto the x- and y-axes. Use interval notation to describe the domain and range of the function.

11. $f(x) = \sqrt{x} + 3$

12. $f(x) = \sqrt{x + 3}$

13. $f(x) = \sqrt{x - 2}$

14. $f(x) = \sqrt{x} - 2$

15. $f(x) = \sqrt{x + 5} + 1$

16. $f(x) = \sqrt{x - 2} - 1$

17. $f(x) = -\sqrt{x + 4}$

18. $f(x) = -\sqrt{x} + 4$

19. $f(x) = -\sqrt{x} + 3$

20. $f(x) = -\sqrt{x + 3}$

---

21. To draw the graph of the function $f(x) = \sqrt{3 - x}$, perform each of the following steps in sequence without the aid of a calculator.

i. Set up a coordinate system and sketch the graph of $y = \sqrt{x}$. Label the graph with its equation.
ii. Set up a second coordinate system and sketch the graph of $y = \sqrt{-x}$. Label the graph with its equation.
iii. Set up a third coordinate system and sketch the graph of $y = \sqrt{-(x - 3)}$. Label the graph with its equation. This is the graph of $f(x) = \sqrt{3 - x}$. Use interval notation to state the domain and range of this function.

22. To draw the graph of the function $f(x) = \sqrt{-x - 3}$, perform each of the following steps in sequence.

i. Set up a coordinate system and sketch the graph of $y = \sqrt{x}$. Label the graph with its equation.
ii. Set up a second coordinate system and sketch the graph of $y = \sqrt{-x}$. Label the graph with its equation.
iii. Set up a third coordinate system and sketch the graph of $y = \sqrt{-(x + 3)}$. Label the graph with its equation. This is the graph of $f(x) = \sqrt{-x - 3}$. Use interval notation to state the domain and range of this function.

23. To draw the graph of the function $f(x) = \sqrt{-x - 1}$, perform each of the following steps in sequence without the aid of a calculator.

i. Set up a coordinate system and sketch the graph of $y = \sqrt{x}$. Label the graph with its equation.
ii. Set up a second coordinate system and sketch the graph of $y = \sqrt{-x}$. Label the graph with its equation.
iii. Set up a third coordinate system and sketch the graph of $y = \sqrt{-(x + 1)}$. Label the graph with its equation. This is the graph of $f(x) = \sqrt{-x - 1}$. Use interval notation to state the domain and range of this function.

24. To draw the graph of the function $f(x) = \sqrt{1-x}$, perform each of the following steps in sequence.

i. Set up a coordinate system and sketch the graph of $y = \sqrt{x}$. Label the graph with its equation.
ii. Set up a second coordinate system and sketch the graph of $y = \sqrt{-x}$. Label the graph with its equation.
iii. Set up a third coordinate system and sketch the graph of $y = \sqrt{-(x-1)}$. Label the graph with its equation. This is the graph of $f(x) = \sqrt{1-x}$. Use interval notation to state the domain and range of this function.

In **Exercises 25-28**, perform each of the following tasks.

i. Draw the graph of the given function with your graphing calculator. Copy the image in your viewing window onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Label your graph with its equation. Use the graph to determine the domain of the function and describe the domain with interval notation.
ii. Use a purely algebraic approach to determine the domain of the given function. Use interval notation to describe your result. Does it agree with the graphical result from part (i)?

25. $f(x) = \sqrt{2x+7}$

26. $f(x) = \sqrt{7-2x}$

27. $f(x) = \sqrt{12-4x}$

28. $f(x) = \sqrt{12+2x}$

In **Exercises 29-40**, find the domain of the given function algebraically.

29. $f(x) = \sqrt{2x+9}$

30. $f(x) = \sqrt{-3x+3}$

31. $f(x) = \sqrt{-8x-3}$

32. $f(x) = \sqrt{-3x+6}$

33. $f(x) = \sqrt{-6x-8}$

34. $f(x) = \sqrt{8x-6}$

35. $f(x) = \sqrt{-7x+2}$

36. $f(x) = \sqrt{8x-3}$

37. $f(x) = \sqrt{6x+3}$

38. $f(x) = \sqrt{x-5}$

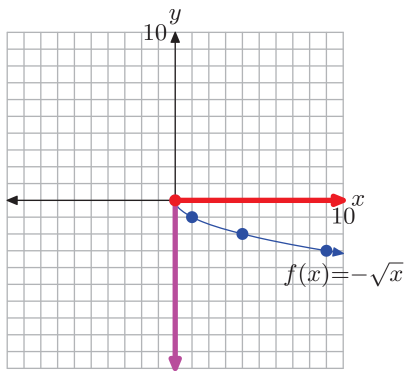
39. $f(x) = \sqrt{-7x-8}$

40. $f(x) = \sqrt{7x+8}$

5.1 Answers

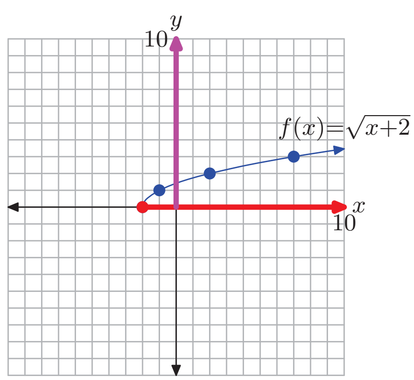
1. Domain $= [0, \infty)$, Range $= (-\infty, 0]$.

x	0	1	4	9
$f(x)$	0	-1	-2	-3



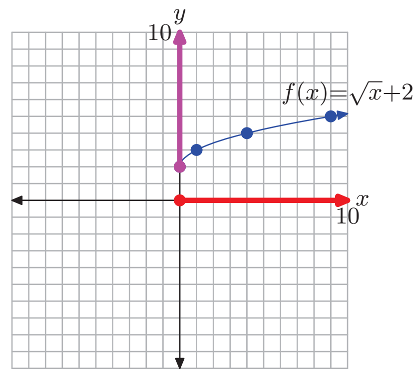
3. Domain $= [-2, \infty)$, Range $= [0, \infty)$.

x	-2	-1	2	7
$f(x)$	0	1	2	3



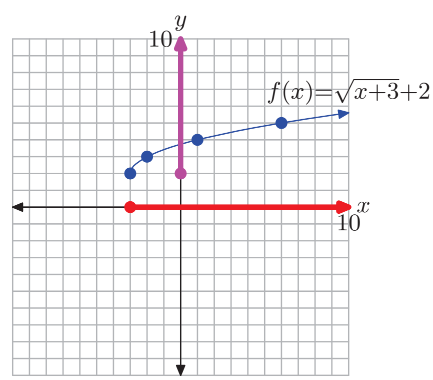
5. Domain $= [0, \infty)$, Range $= [2, \infty)$.

x	0	1	4	9
$f(x)$	2	3	4	5



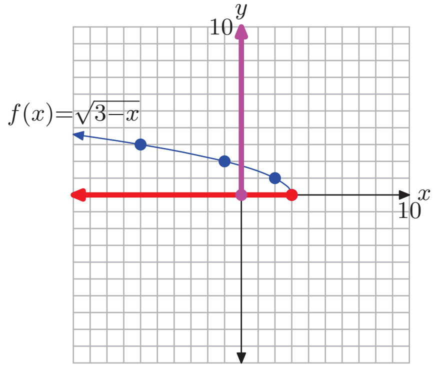
7. Domain $= [-3, \infty)$, Range $= [2, \infty)$.

x	-3	-2	1	6
$f(x)$	2	3	4	5

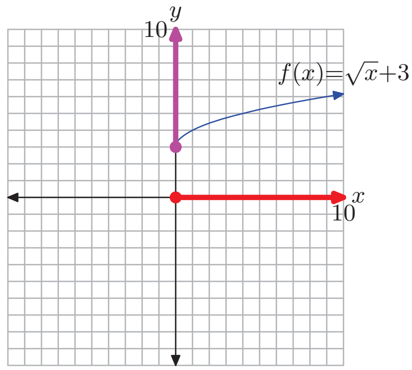


9. Domain $= (-\infty, 3]$, Range $= [0, \infty)$.

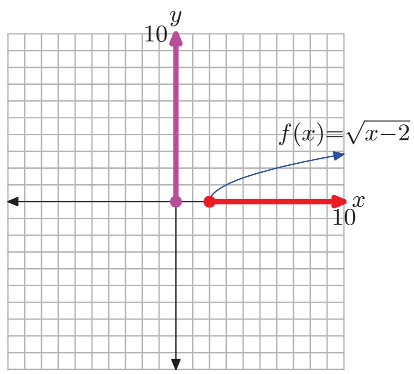
x	-6	-1	2	3
$f(x)$	3	2	1	0



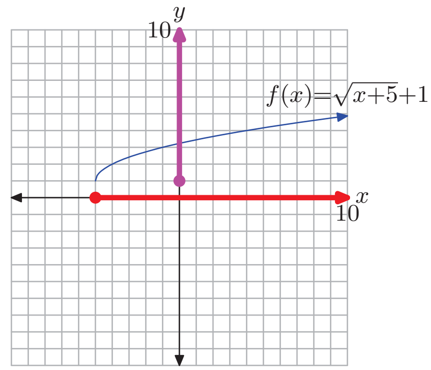
11. Domain $= [0, \infty)$, Range $= [3, \infty)$.



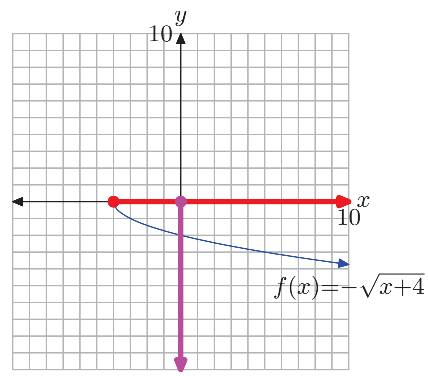
13. Domain $= [2, \infty)$, Range $= [0, \infty)$.



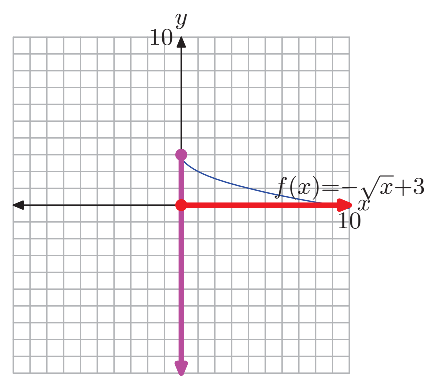
15. Domain $= [-5, \infty)$, Range $= [1, \infty)$.



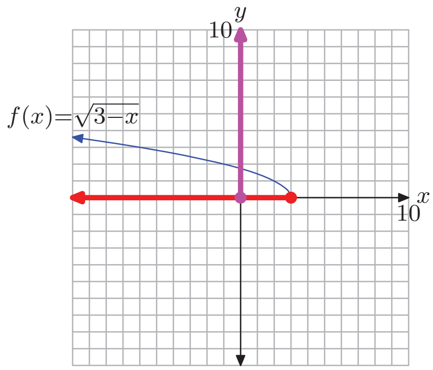
17. Domain $= [-4, \infty)$, Range $= (-\infty, 0]$.



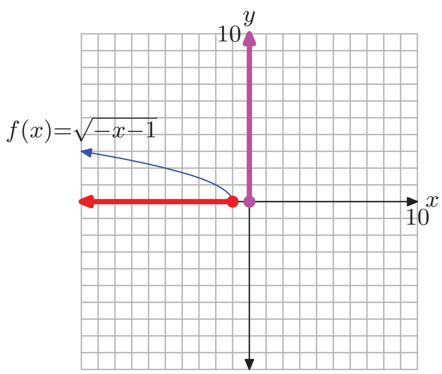
19. Domain $= [0, \infty)$, Range $= (-\infty, 3]$.



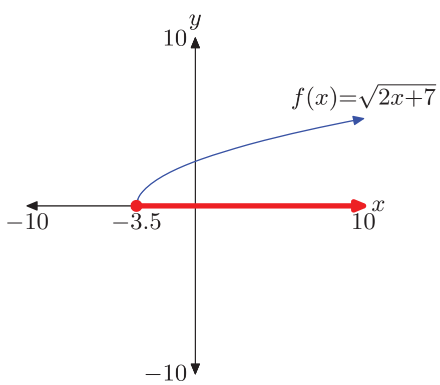
21. Domain $= (-\infty, 3]$, Range $= [0, \infty)$.



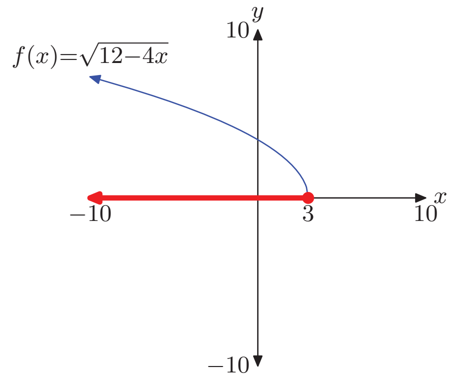
23. Domain $= (-\infty, -1]$, Range $= [0, \infty)$.



25. Domain $= [-7/2, \infty)$



27. Domain $= (-\infty, 3]$



29. $\left[-\frac{9}{2}, \infty\right)$

31. $\left(-\infty, -\frac{3}{8}\right]$

33. $\left(-\infty, -\frac{4}{3}\right]$

35. $\left(-\infty, \frac{2}{7}\right]$

37. $\left[-\frac{1}{2}, \infty\right)$

39. $\left(-\infty, -\frac{8}{7}\right]$

5.2 Multiplication Properties of Radicals

Recall that the equation $x^2 = a$, where a is a positive real number, has two solutions, as indicated in **Figure 1**.

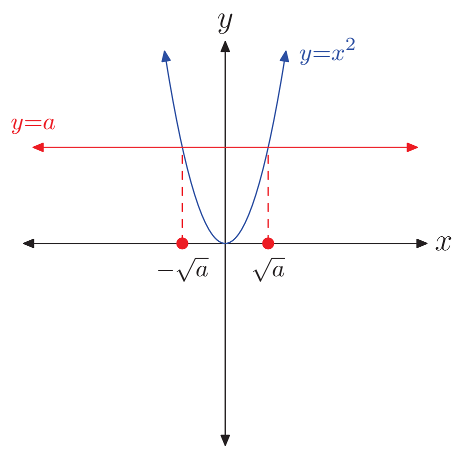


Figure 1. The equation $x^2 = a$, where a is a positive real number, has two solutions.

Here are the key facts.

Solutions of $x^2 = a$. If a is a positive real number, then:

1. The equation $x^2 = a$ has two real solutions.
2. The notation $\sqrt{a}$ denotes the **unique positive** real solution.
3. The notation $-\sqrt{a}$ denotes the **unique negative** real solution.

Note the use of the word **unique**. When we say that $\sqrt{a}$ is the unique positive real solution,[4] we mean that it is the only one. There are no other positive real numbers that are solutions of $x^2 = a$. A similar statement holds for the unique negative solution.

Thus, the equations $x^2 = a$ and $x^2 = b$ have unique positive solutions $x = \sqrt{a}$ and $x = \sqrt{b}$, respectively, provided that a and b are positive real numbers. Furthermore, because they are solutions, they can be substituted into the equations $x^2 = a$ and $x^2 = b$ to produce the results

$$\left(\sqrt{a}\right)^2 = a \qquad \text{and} \qquad \left(\sqrt{b}\right)^2 = b,$$

respectively. Again, these results are dependent upon the fact that a and b are positive real numbers.

Similarly, the equation

[4] Technically, the notation $\sqrt{\ }$ calls for a **nonnegative** real square root, so as to include the possibility $\sqrt{0}$.

$$x^2 = ab$$

has unique positive solution $x = \sqrt{ab}$, provided a and b are positive numbers. However, note that

$$\left(\sqrt{a}\sqrt{b}\right)^2 = \left(\sqrt{a}\right)^2 \left(\sqrt{b}\right)^2 = ab,$$

making $\sqrt{a}\sqrt{b}$ a **second** positive solution of $x^2 = ab$. However, because $\sqrt{ab}$ is the *unique* positive solution of $x^2 = ab$, this forces

$$\sqrt{ab} = \sqrt{a}\sqrt{b}.$$

This discussion leads to the following property of radicals.

Property 1. Let a and b be positive real numbers. Then,

$$\sqrt{ab} = \sqrt{a}\sqrt{b}. \tag{2}$$

This result can be used in two distinctly different ways.

- You can use the result to multiply two square roots, as in

$$\sqrt{7}\sqrt{5} = \sqrt{35}.$$

- You can also use the result to factor, as in

$$\sqrt{35} = \sqrt{5}\sqrt{7}.$$

It is interesting to check this result on the calculator, as shown in **Figure 2**.

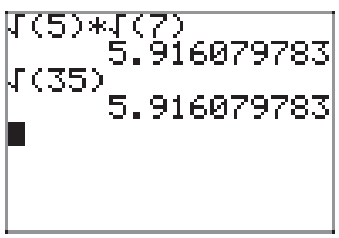


Figure 2. Checking the result $\sqrt{5}\sqrt{7} = \sqrt{35}$.

Simple Radical Form

In this section we introduce the concept of *simple radical form*, but let's first start with a little story. Martha and David are studying together, working a homework problem from their textbook. Martha arrives at an answer of $\sqrt{32}$, while David gets the result $2\sqrt{8}$. At first, David and Martha believe that their solutions are different numbers, but they've been mistaken before so they decide to compare decimal approximations of their results on their calculators. Martha's result is shown in **Figure 3**(a), while David's is shown in **Figure 3**(b).

(a) Martha's result. (b) David's result.

Figure 3. Comparing $\sqrt{32}$ with $2\sqrt{8}$.

Martha finds that $\sqrt{32} \approx 5.656854249$ and David finds that his solution $2\sqrt{8} \approx 5.656854249$. David and Martha conclude that their solutions match, but they want to know why the two very different looking radical expressions are identical.

The following calculation, using **Property 1**, shows why David's result is identical to Martha's.

$$\sqrt{32} = \sqrt{4}\sqrt{8} = 2\sqrt{8}$$

Indeed, there is even a third possibility, one that is much different from the results found by David and Martha. Consider the following calculation, which again uses **Property 1**.

$$\sqrt{32} = \sqrt{16}\sqrt{2} = 4\sqrt{2}$$

In **Figure 4**, note that the decimal approximation of $4\sqrt{2}$ is identical to the decimal approximations for $\sqrt{32}$ (Martha's result in **Figure 3**(a)) and $2\sqrt{8}$ (David's result in **Figure 3**(b)).

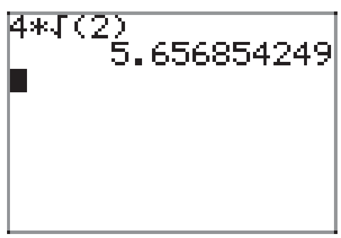


Figure 4. Approximating $4\sqrt{2}$.

While all three of these radical expressions ($\sqrt{32}$, $2\sqrt{8}$, and $4\sqrt{2}$) are identical, it is somewhat frustrating to have so many different forms, particularly when we want to compare solutions. Therefore, we offer a set of guidelines for a special form of the answer which we will call *simple radical form*.

The First Guideline for Simple Radical Form. When possible, factor out a perfect square.

Thus, $\sqrt{32}$ is not in simple radical form, as it is possible to factor out a perfect square, as in

x	x^2
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100
11	121
12	144
13	169
14	196
15	225
16	256
17	289
18	324
19	361
20	400
21	441
22	484
23	529
24	576
25	625

Table 1. Squares.

$$\sqrt{32} = \sqrt{16}\sqrt{2} = 4\sqrt{2}.$$

Similarly, David's result ($2\sqrt{8}$) is not in simple radical form, because he too can factor out a perfect square as follows.

$$2\sqrt{8} = 2(\sqrt{4}\sqrt{2}) = 2(2\sqrt{2}) = (2 \cdot 2)\sqrt{2} = 4\sqrt{2}.$$

If both Martha and David follow the "first guideline for simple radical form," their answers will look identical (both equal $4\sqrt{2}$). This is one of the primary advantages of simple radical form: the ability to compare solutions.

In the examples that follow (and in the exercises), it is helpful if you know the squares of the first 25 positive integers. We've listed them in the margin for you in **Table 1** for future reference.

Let's place a few more radical expressions in simple radical form.

▶ **Example 3.** *Place $\sqrt{50}$ in simple radical form.*

In **Table 1**, 25 is a square. Because $50 = 25 \cdot 2$, we can use **Property 1** to write

$$\sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2}.$$

▶ **Example 4.** *Place $\sqrt{98}$ in simple radical form.*

In **Table 1**, 49 is a square. Because $98 = 49 \cdot 2$, we can again use **Property 1** and write

$$\sqrt{98} = \sqrt{49}\sqrt{2} = 7\sqrt{2}.$$

▶ **Example 5.** *Place $\sqrt{288}$ in simple radical form.*

Some students seem able to pluck the optimal "perfect square" out of thin air. If you consult **Table 1**, you'll note that 144 is a square. Because $288 = 144 \cdot 2$, we can write

$$\sqrt{288} = \sqrt{144}\sqrt{2} = 12\sqrt{2}.$$

However, what if you miss that higher perfect square, think $288 = 4 \cdot 72$, and write

$$\sqrt{288} = \sqrt{4}\sqrt{72} = 2\sqrt{72}.$$

This approach is not incorrect, provided you realize that you're not finished. You can still factor a perfect square out of 72. Because $72 = 36 \cdot 2$, you can continue and write

$$2\sqrt{72} = 2(\sqrt{36}\sqrt{2}) = 2(6\sqrt{2}) = (2 \cdot 6)\sqrt{2} = 12\sqrt{2}.$$

Note that we arrived at the same simple radical form, namely $12\sqrt{2}$. It just took us a little longer. As long as we realize that we must continue until we can no longer factor out a perfect square, we'll arrive at the same simple radical form as the student who seems to magically pull the higher square out of thin air.

Indeed, here is another approach that is equally valid.

$$\sqrt{288} = \sqrt{4}\sqrt{72} = 2(\sqrt{4}\sqrt{18}) = 2(2\sqrt{18}) = (2 \cdot 2)\sqrt{18} = 4\sqrt{18}$$

We need to recognize that we are still not finished because we can extract another perfect square as follows.

$$4\sqrt{18} = 4(\sqrt{9}\sqrt{2}) = 4(3\sqrt{2}) = (4 \cdot 3)\sqrt{2} = 12\sqrt{2}$$

Once again, same result. However, note that it behooves us to extract the largest square possible, as it minimizes the number of steps required to attain simple radical form.

Checking Results with the Graphing Calculator. Once you've placed a radical expression in simple radical form, you can use your graphing calculator to check your result. In this example, we found that

$$\sqrt{288} = 12\sqrt{2}. \tag{6}$$

Enter the left- and right-hand sides of this result as shown in **Figure 5**. Note that each side produces the same decimal approximation, verifying the result in **equation (6)**.

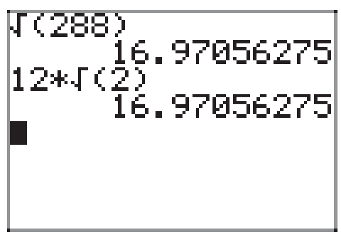


Figure 5. Comparing $\sqrt{288}$ with its simple radical form $12\sqrt{2}$.

Helpful Hints

Recall that raising a power of a base to another power requires that we multiply exponents.

Raising a Power of a Base to another Power.

$$(a^m)^n = a^{mn}$$

In particular, when you square a power of a base, you must multiply the exponent by 2. For example,

$$(2^5)^2 = 2^{10}.$$

Conversely, because taking a square root is the "inverse" of squaring,[5] when taking a square root we must *divide* the existing exponent by 2, as in

$$\sqrt{2^{10}} = 2^5.$$

Note that squaring 2^5 gives 2^{10}, so taking the square root of 2^{10} must return you to 2^5. When you square, you double the exponent. Therefore, when you take the square root, you must halve the exponent.

Similarly,

- $(2^6)^2 = 2^{12}$ so $\sqrt{2^{12}} = 2^6$.
- $(2^7)^2 = 2^{14}$ so $\sqrt{2^{14}} = 2^7$.
- $(2^8)^2 = 2^{16}$ so $\sqrt{2^{16}} = 2^8$.

This leads to the following result.

Taking the Square Root of an Even Power. When taking a square root of x^n, when x is a positive real number and n is an even natural number, divide the exponent by two. In symbols,

$$\sqrt{x^n} = x^{n/2}.$$

Note that this agrees with the definition of rational exponents presented in Chapter 8, as in

$$\sqrt{x^n} = (x^n)^{1/2} = x^{n/2}.$$

On another note, recall that raising a product to a power requires that we raise each factor to that power.

Raising a Product to a Power.

$$(ab)^n = a^n b^n.$$

In particular, if you square a product, you must square each factor. For example,

$$(5^3 7^4)^2 = (5^3)^2 (7^4)^2 = 5^6 7^8.$$

Note that we multiplied each existing exponent in this product by 2.

[5] Well, not always. Consider $(-2)^2 = 4$, but $\sqrt{4} = 2$ does not return to -2. However, when you start with a postive number and square, then taking the positive square root is the inverse operation and returns you to the original positive number. Return to Chapter 8 (the section on inverse functions) if you want to reread a full discussion of this trickiness.

Property 1 is similar, in that when we take the square root of a product, we take the square root of each factor. Because taking a square root is the inverse of squaring, we must *divide* each existing exponent by 2, as in

$$\sqrt{5^6 7^8} = \sqrt{5^6}\sqrt{7^8} = 5^3 7^4.$$

Let's look at some examples that employ this technique.

▶ **Example 7.** *Simplify* $\sqrt{2^4 3^6 5^{10}}$.

When taking the square root of a product of exponential factors, divide each exponent by 2.

$$\sqrt{2^4 3^6 5^{10}} = 2^2 3^3 5^5$$

If needed, you can expand the exponential factors and multiply to provide a single numerical answer.

$$2^2 3^3 5^5 = 4 \cdot 27 \cdot 3125 = 337\,500$$

A calculator was used to obtain the final solution.

▶ **Example 8.** *Simplify* $\sqrt{2^5 3^3}$.

In this example, the difficulty is the fact that the exponents are not divisible by 2. However, if possible, the "first guideline of simple radical form" requires that we factor out a perfect square. So, extract each factor raised to the highest possible power that is divisible by 2, as in

$$\sqrt{2^5 3^3} = \sqrt{2^4\ 3^2}\sqrt{2 \cdot 3}$$

Now, divide each exponent by 2.

$$\sqrt{2^4\ 3^2}\sqrt{2 \cdot 3} = 2^2 3^1 \sqrt{2 \cdot 3}$$

Finally, simplify by expanding each exponential factor and multiplying.

$$2^2 3^1 \sqrt{2 \cdot 3} = 4 \cdot 3\sqrt{2 \cdot 3} = 12\sqrt{6}$$

▶ **Example 9.** *Simplify* $\sqrt{3^7 5^2 7^5}$.

Extract each factor to the highest possible power that is divisible by 2.

$$\sqrt{3^7 5^2 7^5} = \sqrt{3^6 5^2 7^4}\sqrt{3 \cdot 7}$$

Divide each exponent by 2.

$$\sqrt{3^6 5^2 7^4}\sqrt{3 \cdot 7} = 3^3 5^1 7^2 \sqrt{3 \cdot 7}$$

Expand each exponential factor and multiply.

$$3^3 5^1 7^2 \sqrt{3 \cdot 7} = 27 \cdot 5 \cdot 49 \sqrt{3 \cdot 7} = 6\,615\sqrt{21}$$

▶ **Example 10.** *Place* $\sqrt{216}$ *in simple radical form.*

If we prime factor 216, we can attack this problem with the same technique used in the previous examples. Before we prime factor 216, here are a few divisibility tests that you might find useful.

Divisibility Tests.

- If a number ends in 0, 2, 4, 6, or 8, it is an **even** number and is divisible by 2.
- If the last two digits of a number form a number that is divisible by 4, then the entire number is divisible by 4.
- If a number ends in 0 or 5, it is divisible by 5.
- If the sum of the digits of a number is divisible by 3, then the entire number is divisible by 3.
- If the sum of the digits of a number is divisible by 9, then the entire number is divisible by 9.

For example, in order:

- The number 226 ends in a 6, so it is even and divisible by 2. Indeed, $226 = 2 \cdot 113$.
- The last two digits of 224 are 24, which is divisible by 4, so the entire number is divisible by 4. Indeed, $224 = 4 \cdot 56$.
- The last digit of 225 is a 5. Therefore 225 is divisible by 5. Indeed, $225 = 5 \cdot 45$.
- The sum of the digits of 222 is $2+2+2 = 6$, which is divisible by 3. Therefore, 222 is divisible by 3. Indeed, $222 = 3 \cdot 74$.
- The sum of the digits of 684 is $6 + 8 + 4 = 18$, which is divisible by 9. Therefore, 684 is divisible by 9. Indeed, $684 = 9 \cdot 76$.

Now, let's prime factor 216. Note that $2+1+6 = 9$, so 216 is divisible by 9. Indeed, $216 = 9 \cdot 24$. In **Figure 6**, we use a "factor tree" to continue factoring until all of the "leaves" are prime numbers.

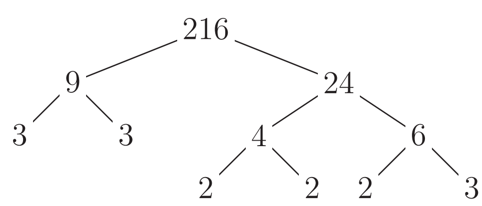


Figure 6. Using a factor tree to prime factor 216.

Thus,

$$216 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3,$$

or in exponential form,

$$216 = 2^3 \cdot 3^3.$$

Thus,

$$\sqrt{216} = \sqrt{2^3\,3^3} = \sqrt{2^2\,3^2}\,\sqrt{2 \cdot 3} = 2 \cdot 3\sqrt{2 \cdot 3} = 6\sqrt{6}.$$

Prime factorization is an unbelievably useful tool!

Let's look at another example.

► **Example 11.** *Place* $\sqrt{2592}$ *in simple radical form.*

If we find the prime factorization for 2592, we can attack this example using the same technique we used in the previous example. We note that the sum of the digits of 2592 is $2 + 5 + 9 + 2 = 18$, which is divisible by 9. Therefore, 2592 is also divisible by 9.

$$2592 = 9 \cdot 288$$

The sum of the digits of 288 is $2 + 8 + 8 = 18$, which is divisible by 9, so 288 is also divisible by 9.

$$2592 = 9 \cdot (9 \cdot 32)$$

Continue in this manner until the leaves of your "factor tree" are all primes. Then, you should get

$$2592 = 2^5\,3^4.$$

Thus,

$$\sqrt{2592} = \sqrt{2^5 3^4} = \sqrt{2^4 3^4}\sqrt{2} = 2^2 3^2\sqrt{2} = 4 \cdot 9\sqrt{2} = 36\sqrt{2}.$$

Let's use the graphing calculator to check this result. Enter each side of $\sqrt{2592} = 36\sqrt{2}$ separately and compare approximations, as shown in **Figure 7**.

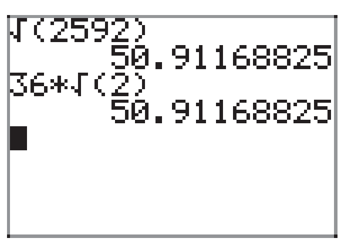


Figure 7. Comparing $\sqrt{2592}$ with its simple radical form $36\sqrt{2}$.

An Important Property of Square Roots

One of the most common mistakes in algebra occurs when practitioners are asked to simplify the expression $\sqrt{x^2}$, where x is any arbitrary real number. Let's examine two of the most common errors.

- Some will claim that the following statement is true for any arbitrary real number x.

  $$\sqrt{x^2} = \pm x.$$

  This is easily seen to be incorrect. Simply substitute any real number for x to check this claim. We will choose $x = 3$ and substitute it into each side of the proposed statement.

  $$\sqrt{3^2} = \pm 3$$

  If we simplify the left-hand side, we produce the following result.

  $$\begin{aligned} \sqrt{3^2} &= \pm 3 \\ 3 &= \pm 3 \end{aligned}$$

  It is not correct to state that 3 and ± 3 are equal.
- A second error is to claim that

  $$\sqrt{x^2} = x$$

  for any arbitrary real number x. Although this is certainly true if you substitute nonnegative numbers for x, look what happens when you substitute -3 for x.

  $$\sqrt{(-3)^2} = -3$$

  If we simplify the left-hand side, we produce the following result.

  $$\begin{aligned} \sqrt{9} &= -3 \\ 3 &= -3 \end{aligned}$$

  Clearly, 3 and -3 are not equal.

In both cases, what has been forgotten is the fact that $\sqrt{}$ calls for a positive (nonnegative if you want to include the case $\sqrt{0}$) square root. In both of the errors above, namely $\sqrt{x^2} = \pm x$ and $\sqrt{x^2} = x$, the left-hand side is calling for a nonnegative response, but nothing has been done to insure that the right-hand side is also nonnegative. Does anything come to mind?

Sure, if we wrap the right-hand side in absolute values, as in

$$\sqrt{x^2} = |x|,$$

then both sides are calling for a nonnegative response. Indeed, note that

$$\sqrt{(-3)^2} = |-3|, \qquad \sqrt{0^2} = |0|, \qquad \text{and} \qquad \sqrt{3^2} = |3|$$

are all valid statements.

This discussion leads to the following result.

The Positive Square Root of the Square of x. If x is any real number, then

$$\sqrt{x^2} = |x|.$$

The next task is to use this new property to produce a extremely useful property of absolute value.

A Multiplication Property of Absolute Value

If we combine the law of exponents for squaring a product with our property for taking the square root of a product, we can write

$$\sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2}\sqrt{b^2}.$$

However, $\sqrt{(ab)^2} = |ab|$, while $\sqrt{a^2}\sqrt{b^2} = |a||b|$. This discussion leads to the following result.

Product Rule for Absolute Value. If a and b are any real numbers,

$$|ab| = |a||b|. \tag{12}$$

In words, the absolute value of a product is equal to the product of the absolute values.

We saw this property previously in the chapter on the absolute value function, where we provided a different approach to the proof of the property. It's interesting that we can prove this property in a completely new way using the properties of square root. We'll see we have need for the Product Rule for Absolute Value in the examples that follow.

For example, using the product rule, if x is any real number, we could write

$$|3x| = |3||x| = 3|x|$$

However, there is no way we can remove the absolute value bars that surround x unless we know the sign of x. If $x \geq 0$, then $|x| = x$ and the expression becomes

$$3|x| = 3x.$$

On the other hand, if $x < 0$, then $|x| = -x$ and the expression becomes

$$3|x| = 3(-x) = -3x.$$

Let's look at another example. Using the product rule, if x is any real number, the expression $|-4x^3|$ can be manipulated as follows.

$$|-4x^3| = |-4||x^2||x|$$

However, $|-4| = 4$ and since $x^2 \geq 0$ for any value of x, $|x^2| = x^2$. Thus,

$$|-4||x^2||x| = 4x^2|x|.$$

Again, there is no way we can remove the absolute value bars around x unless we know the sign of x. If $x \geq 0$, then $|x| = x$ and

$$4x^2|x| = 4x^2(x) = 4x^3.$$

On the other hand, if $x < 0$, then $|x| = -x$ and

$$4x^2|x| = 4x^2(-x) = -4x^3.$$

Let's use these ideas to simplify some radical expressions that contain variables.

Variable Expressions

▶ **Example 13.** *Given that the x represents any real numbers, place the radical expression*

$$\sqrt{48x^6}$$

in simple radical form.

Simple radical form demands that we factor out a perfect square, if possible. In this case, $48 = 16 \cdot 3$ and we factor out the highest power of x that is divisible by 2.

$$\sqrt{48x^6} = \sqrt{16x^6}\sqrt{3}$$

We can now use **Property 1** to take the square root of each factor.

$$\sqrt{16x^6}\sqrt{3} = \sqrt{16}\sqrt{x^6}\sqrt{3}$$

Now, remember that the notation $\sqrt{\ }$ calls for a **nonnegative** square root, so we must insure that each response in the equation above is nonnegative. Thus,

$$\sqrt{16}\sqrt{x^6}\sqrt{3} = 4|x^3|\sqrt{3}.$$

Some comments are in order.

- The nonnegative square root of 16 is 4. That is, $\sqrt{16} = 4$.
- The nonnegative square root of x^6 is trickier. It is incorrect to say $\sqrt{x^6} = x^3$, because x^3 could be negative (if x is negative). To insure a nonnegative square root, in this case we need to wrap our answer in absolute value bars. That is, $\sqrt{x^6} = |x^3|$.

We can use the Product Rule for Absolute Value to write $|x^3| = |x^2||x|$. Because x^2 is nonnegative, absolute value bars are redundant and not needed. That is, $|x^2||x| = x^2|x|$. Thus, we can simplify our solution a bit further and write

$$4|x^3|\sqrt{3} = 4x^2|x|\sqrt{3}.$$

Thus,

$$\sqrt{48x^6} = 4x^2|x|\sqrt{3}. \tag{14}$$

Alternate Solution. There is a variety of ways that we can place a radical expression in simple radical form. Here is another approach. Starting at the step above, where we first factored out a perfect square,

$$\sqrt{48x^6} = \sqrt{16x^6}\sqrt{3},$$

we could write

$$\sqrt{16x^6}\sqrt{3} = \sqrt{(4x^3)^2}\sqrt{3}.$$

Now, remember that the nonnegative square root of the square of an expression is the absolute value of that expression (we have to guarantee a nonnegative answer), so

$$\sqrt{(4x^3)^2}\sqrt{3} = |4x^3|\sqrt{3}.$$

However, $|4x^3| = |4||x^3|$ by our product rule and $|4||x^3| = 4|x^3|$. Thus,

$$|4x^3|\sqrt{3} = 4|x^3|\sqrt{3}.$$

Finally, $|x^3| = |x^2||x| = x^2|x|$ because $x^2 \geq 0$, so we can write

$$4|x^3|\sqrt{3} = 4x^2|x|\sqrt{3}. \tag{15}$$

We cannot remove the absolute value bar that surrounds x unless we know the sign of x.

Note that the simple radical form (**15**) in the alternate solution is identical to the simple radical form (**14**) found with the previous solution technique.

Let's look at another example.

▶ **Example 16.** *Given that $x < 0$, place $\sqrt{24x^6}$ in simple radical form.*

First, factor out a perfect square and write

$$\sqrt{24x^6} = \sqrt{4x^6}\sqrt{6}.$$

Now, use **Property 1** and take the square root of each factor.

$$\sqrt{4x^6}\sqrt{6} = \sqrt{4}\sqrt{x^6}\sqrt{6}$$

To insure a nonnegative response to $\sqrt{x^6}$, wrap your response in absolute values.

$$\sqrt{4}\sqrt{x^6}\sqrt{6} = 2|x^3|\sqrt{6}$$

However, as in the previous problem, $|x^3| = |x^2||x| = x^2|x|$, since $x^2 \geq 0$. Thus,

$$2|x^3|\sqrt{6} = 2x^2|x|\sqrt{6}.$$

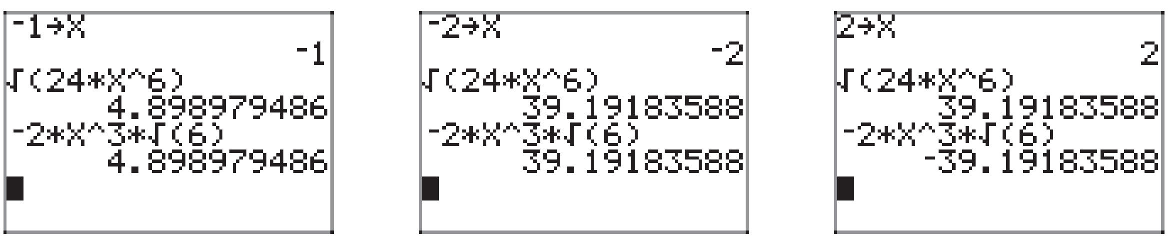
In this example, we were given the extra fact that $x < 0$, so $|x| = -x$ and we can write

$$2x^2|x|\sqrt{6} = 2x^2(-x)\sqrt{6} = -2x^3\sqrt{6}.$$

It is instructive to test the validity of the answer

$$\sqrt{24x^6} = -2x^3\sqrt{6}, \quad x < 0,$$

using a calculator. So, set $x = -1$ with the command `-1 STO▶X`. That is, enter -1, then push the `STO▶` button, followed by `X`, then press the `ENTER` key. The result is shown in **Figure 8**(a). Next, enter `√(24*X^6)` and press `ENTER` to capture the second result shown in **Figure 8**(a). Finally, enter `-2*X^3√(6)` and press `ENTER`. Note that the two expressions $\sqrt{24x^6}$ and $-2x^3\sqrt{6}$ agree at $x = -1$, as seen in **Figure 8**(a). We've also checked the validity of the result at $x = -2$ in **Figure 8**(b). However, note that our result is not valid at $x = 2$ in **Figure 8**(c). This occurs because $\sqrt{24x^6} = -2x^3\sqrt{6}$ only if x is negative.



(a) Check with $x = -1$. (b) Check with $x = -2$. (c) Check with $x = 2$.

Figure 8. Spot-checking the validity of $\sqrt{24x^6} = -2x^3\sqrt{6}$.

It is somewhat counterintuitive that the result

$$\sqrt{24x^6} = -2x^3\sqrt{6}, \quad x < 0,$$

contains a negative sign. After all, the expression $\sqrt{24x^6}$ calls for a nonnegative result, but we have a negative sign. However, on closer inspection, if $x < 0$, then x is a negative number and the right-hand side $-2x^3\sqrt{6}$ is a positive number (-2 is negative, x^3 is negative because x is negative, and the product of two negatives is a positive).

Let's look at another example.

▶ **Example 17.** *If $x < 3$, simplify $\sqrt{x^2 - 6x + 9}$.*

The expression under the radical is a perfect square trinomial and factors.

$$\sqrt{x^2 - 6x + 9} = \sqrt{(x-3)^2}$$

However, the nonnegative square root of the square of an expression is the absolute value of that expression, so

$$\sqrt{(x-3)^2} = |x-3|.$$

Finally, because we are told that $x < 3$, this makes $x - 3$ a negative number, so

$$|x-3| = -(x-3). \tag{18}$$

Again, the result $\sqrt{x^2 - 6x + 9} = -(x-3)$, provided $x < 3$, is somewhat counterintuitive as we are expecting a positive result. However, if $x < 3$, the result $-(x-3)$ is positive. You can test this by substituting several values of x that are less than 3 into the expression $-(x-3)$ and noting that the result is positive. For example, if $x = 2$, then x is less than 3 and

$$-(x-3) = -(2-3) = -(-1) = 1,$$

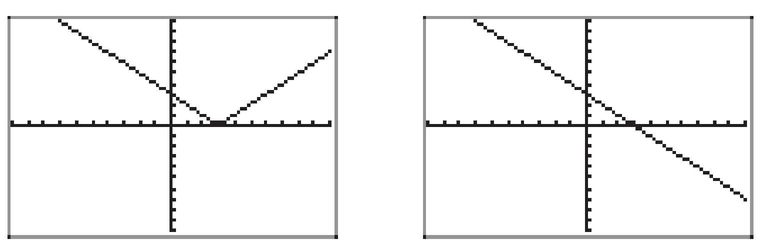
which, of course, is a positive result.

It is even more informative to note that our result is equivalent to

$$\sqrt{x^2 - 6x + 9} = -x + 3, \quad x < 3.$$

This is easily seen by distributing the minus sign in the result (**18**).

We've drawn the graph of $y = \sqrt{x^2 - 6x + 9}$ on our calculator in **Figure 9**(a). In **Figure 9**(b), we've drawn the graph of $y = -x + 3$. Note that the graphs agree when $x < 3$. Indeed, when you consider the left-hand branch of the "V" in **Figure 9**(a), you can see that the slope of this branch is -1 and the y-intercept is 3. The equation of this branch is $y = -x + 3$, so it agrees with the graph of $y = -x + 3$ in **Figure 9**(b) when x is less than 3.



(a) The graph of $y = \sqrt{x^2 - 6x + 9}$.

(b) The graph of $y = -x + 3$.

Figure 9. Verifying graphically that $\sqrt{x^2 - 6x + 9} = -x + 3$ when $x < 3$.

5.2 Exercises

1. Use a calculator to first approximate $\sqrt{5}\sqrt{2}$. On the same screen, approximate $\sqrt{10}$. Report the results on your homework paper.

2. Use a calculator to first approximate $\sqrt{7}\sqrt{10}$. On the same screen, approximate $\sqrt{70}$. Report the results on your homework paper.

3. Use a calculator to first approximate $\sqrt{3}\sqrt{11}$. On the same screen, approximate $\sqrt{33}$. Report the results on your homework paper.

4. Use a calculator to first approximate $\sqrt{5}\sqrt{13}$. On the same screen, approximate $\sqrt{65}$. Report the results on your homework paper.

In **Exercises 5-20**, place each of the radical expressions in simple radical form. As in Example 3 in the narrative, check your result with your calculator.

5. $\sqrt{18}$

6. $\sqrt{80}$

7. $\sqrt{112}$

8. $\sqrt{72}$

9. $\sqrt{108}$

10. $\sqrt{54}$

11. $\sqrt{50}$

12. $\sqrt{48}$

13. $\sqrt{245}$

14. $\sqrt{150}$

15. $\sqrt{98}$

16. $\sqrt{252}$

17. $\sqrt{45}$

18. $\sqrt{294}$

19. $\sqrt{24}$

20. $\sqrt{32}$

In **Exercises 21-26**, use prime factorization (as in Examples 10 and 11 in the narrative) to assist you in placing the given radical expression in simple radical form. Check your result with your calculator.

21. $\sqrt{2016}$

22. $\sqrt{2700}$

23. $\sqrt{14175}$

24. $\sqrt{44000}$

25. $\sqrt{20250}$

26. $\sqrt{3564}$

In **Exercises 27-46**, place each of the given radical expressions in simple radical form. Make no assumptions about the sign of the variables. Variables can either represent positive or negative numbers.

27. $\sqrt{(6x-11)^4}$

28. $\sqrt{16h^8}$

29. $\sqrt{25f^2}$

30. $\sqrt{25j^8}$

31. $\sqrt{16m^2}$

32. $\sqrt{25a^2}$

33. $\sqrt{(7x+5)^{12}}$

34. $\sqrt{9w^{10}}$

35. $\sqrt{25x^2-50x+25}$

36. $\sqrt{49x^2-42x+9}$

37. $\sqrt{25x^2+90x+81}$

38. $\sqrt{25f^{14}}$

39. $\sqrt{(3x+6)^{12}}$

40. $\sqrt{(9x-8)^{12}}$

41. $\sqrt{36x^2+36x+9}$

42. $\sqrt{4e^2}$

43. $\sqrt{4p^{10}}$

44. $\sqrt{25x^{12}}$

45. $\sqrt{25q^6}$

46. $\sqrt{16h^{12}}$

---

47. Given that $x < 0$, place the radical expression $\sqrt{32x^6}$ in simple radical form. Check your solution on your calculator for $x = -2$.

48. Given that $x < 0$, place the radical expression $\sqrt{54x^8}$ in simple radical form. Check your solution on your calculator for $x = -2$.

49. Given that $x < 0$, place the radical expression $\sqrt{27x^{12}}$ in simple radical form. Check your solution on your calculator for $x = -2$.

50. Given that $x < 0$, place the radical expression $\sqrt{44x^{10}}$ in simple radical form. Check your solution on your calculator for $x = -2$.

In **Exercises 51-54**, follow the lead of Example 17 in the narrative to simplify the given radical expression and check your result with your graphing calculator.

51. Given that $x < 4$, place the radical expression $\sqrt{x^2-8x+16}$ in simple radical form. Use a graphing calculator to show that the graphs of the original expression and your simple radical form agree for all values of x such that $x < 4$.

52. Given that $x \geq -2$, place the radical expression $\sqrt{x^2+4x+4}$ in simple radical form. Use a graphing calculator to show that the graphs of the original expression and your simple radical form agree for all values of x such that $x \geq -2$.

53. Given that $x \geq 5$, place the radical expression $\sqrt{x^2-10x+25}$ in simple radical form. Use a graphing calculator to show that the graphs of the original expression and your simple radical form agree for all values of x such that $x \geq 5$.

54. Given that $x < -1$, place the radical expression $\sqrt{x^2+2x+1}$ in simple radical form. Use a graphing calculator to show that the graphs of the original expression and your simple radical form agree for all values of x such that $x < -1$.

In **Exercises 55-72**, place each radical expression in simple radical form. Assume that all variables represent positive numbers.

55. $\sqrt{9d^{13}}$

56. $\sqrt{4k^2}$

57. $\sqrt{25x^2+40x+16}$

58. $\sqrt{9x^2-30x+25}$

59. $\sqrt{4j^{11}}$

60. $\sqrt{16j^6}$

61. $\sqrt{25m^2}$

62. $\sqrt{9e^9}$

63. $\sqrt{4c^5}$

64. $\sqrt{25z^2}$

65. $\sqrt{25h^{10}}$

66. $\sqrt{25b^2}$

67. $\sqrt{9s^7}$

68. $\sqrt{9e^7}$

69. $\sqrt{4p^8}$

70. $\sqrt{9d^{15}}$

71. $\sqrt{9q^{10}}$

72. $\sqrt{4w^7}$

In **Exercises 73-80**, place each given radical expression in simple radical form. Assume that all variables represent positive numbers.

73. $\sqrt{2f^5}\sqrt{8f^3}$

74. $\sqrt{3s^3}\sqrt{243s^3}$

75. $\sqrt{2k^7}\sqrt{32k^3}$

76. $\sqrt{2n^9}\sqrt{8n^3}$

77. $\sqrt{2e^9}\sqrt{8e^3}$

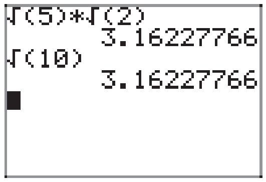
78. $\sqrt{5n^9}\sqrt{125n^3}$

79. $\sqrt{3z^5}\sqrt{27z^3}$

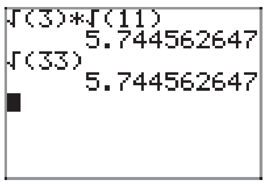
80. $\sqrt{3t^7}\sqrt{27t^3}$

5.2 Answers

1.



3.



5. $3\sqrt{2}$

7. $4\sqrt{7}$

9. $6\sqrt{3}$

11. $5\sqrt{2}$

13. $7\sqrt{5}$

15. $7\sqrt{2}$

17. $3\sqrt{5}$

19. $2\sqrt{6}$

21. $12\sqrt{14}$

23. $45\sqrt{7}$

25. $45\sqrt{10}$

27. $(6x-11)^2$

29. $5|f|$

31. $4|m|$

33. $(7x+5)^6$

35. $|5x-5|$

37. $|5x+9|$

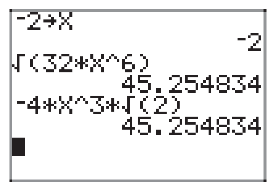
39. $(3x+6)^6$

41. $|6x+3|$

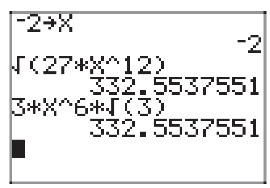
43. $2p^4|p|$

45. $5q^2|q|$

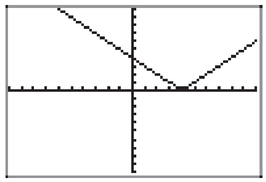
47. $-4x^3\sqrt{2}$

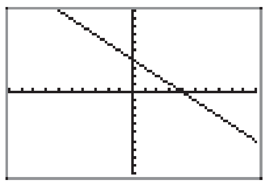


49. $3x^6\sqrt{3}$

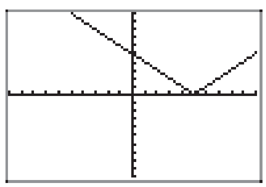


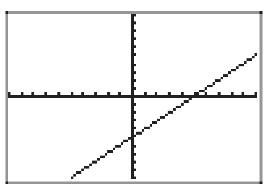
51. $-x+4$. The graphs of $y=-x+4$ and $y=\sqrt{x^2-8x+16}$ follow. Note that they agree for $x<4$.





53. $x-5$. The graphs of $y=x-5$ and $y=\sqrt{x^2-10x+25}$ follow. Note that they agree for $x\geq 5$.





55. $3d^6\sqrt{d}$

57. $5x+4$

59. $2j^5\sqrt{j}$

61. $5m$

63. $2c^2\sqrt{c}$

65. $5h^5$

67. $3s^3\sqrt{s}$

69. $2p^4$

71. $3q^5$

73. $4f^4$

75. $8k^5$

77. $4e^6$

79. $9z^4$

5.3 Division Properties of Radicals

Each of the equations $x^2 = a$ and $x^2 = b$ has a unique positive solution, $x = \sqrt{a}$ and $x = \sqrt{b}$, respectively, provided a and b are positive real numbers. Further, because they are solutions, they can be substituted into the equations $x^2 = a$ and $x^2 = b$ to produce the results

$$(\sqrt{a})^2 = a \qquad \text{and} \qquad (\sqrt{b})^2 = b,$$

respectively. These results are dependent upon the fact that a and b are positive real numbers.

Similarly, the equation

$$x^2 = \frac{a}{b}$$

has the unique positive solution

$$x = \sqrt{\frac{a}{b}},$$

provided a and b are positive real numbers. However, note that

$$\left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 = \frac{(\sqrt{a})^2}{(\sqrt{b})^2} = \frac{a}{b},$$

making $\sqrt{a}/\sqrt{b}$ a **second** positive solution of $x^2 = a/b$. However, because $\sqrt{a/b}$ is the *unique* positive solution of $x^2 = a/b$, this forces

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

This discussion leads us to the following property of radicals.

Property 1. Let a and b be positive real numbers. Then,

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

This result can be used in two distinctly different ways.

- You can use the result to divide two square roots, as in

$$\frac{\sqrt{13}}{\sqrt{7}} = \sqrt{\frac{13}{7}}.$$

[7] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

- You can also use the result to take the square root of a fraction. Simply take the square root of both numerator and denominator, as in

$$\sqrt{\frac{13}{7}} = \frac{\sqrt{13}}{\sqrt{7}}.$$

It is interesting to check these results on a calculator, as shown in **Figure 1**.

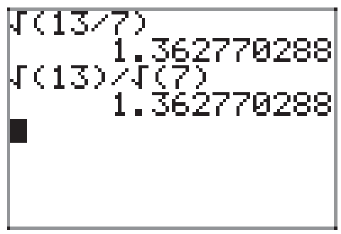
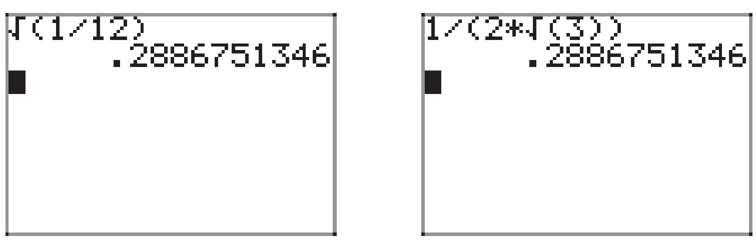


Figure 1. Checking that $\sqrt{13/7} = \sqrt{13}/\sqrt{7}$.

Simple Radical Form Continued

David and Martha are again working on a homework problem. Martha obtains the solution $\sqrt{1/12}$, but David's solution $1/(2\sqrt{3})$ is seemingly different. Having learned their lesson in an earlier assignment, they use their calculators to find decimal approximations of their solutions. Martha's approximation is shown in **Figure 2**(a) and David's approximation is shown in **Figure 2**(b).



(a) Approximating Martha's $\sqrt{1/12}$.

(b) Approximating David's $1/(2\sqrt{3})$.

Figure 2. Comparing Martha's $\sqrt{1/12}$ with David's $1/(2\sqrt{3})$.

Martha finds that $\sqrt{1/12} \approx 0.2886751346$ and David finds that $1/(2\sqrt{3}) \approx 0.2886751346$. They conclude that their answers match, but they want to know why such different looking answers are identical.

The following calculation shows why Martha's result is identical to David's. First, use the division property of radicals (**Property 1**) to take the square root of both numerator and denominator.

$$\sqrt{\frac{1}{12}} = \frac{\sqrt{1}}{\sqrt{12}} = \frac{1}{\sqrt{12}}$$

Next, use the "first guideline for simple radical form" and factor a perfect square from the denominator.

$$\frac{1}{\sqrt{12}} = \frac{1}{\sqrt{4}\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

This clearly demonstrates that David and Martha's solutions are identical.

Indeed, there are other possible forms for the solution of David and Martha's homework exercise. Start with Martha's solution, then multiply both numerator and denominator of the fraction under the radical by 3.

$$\sqrt{\frac{1}{12}} = \sqrt{\frac{1}{12} \cdot \frac{3}{3}} = \sqrt{\frac{3}{36}}$$

Now, use the division property of radicals (**Property 1**), taking the square root of both numerator and denominator.

$$\sqrt{\frac{3}{36}} = \frac{\sqrt{3}}{\sqrt{36}} = \frac{\sqrt{3}}{6}$$

Note that the approximation of $\sqrt{3}/6$ in **Figure 3** is identical to Martha's and David's approximations in **Figures 2**(a) and (b).

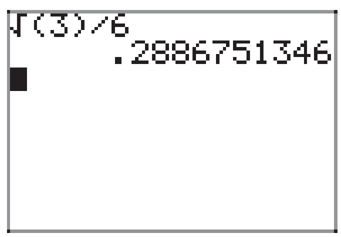


Figure 3. Finding an approximation of $\sqrt{3}/6$.

While all three of the solution forms ($\sqrt{1/12}$, $1/(2\sqrt{3})$, and $\sqrt{3}/6$) are identical, it is very frustrating to have so many forms, particularly when we want to compare solutions. So, we are led to establish two more guidelines for simple radical form.

The Second Guideline for Simple Radical Form. Don't leave fractions under a radical.

Thus, Martha's $\sqrt{1/12}$ is not in simple radical form, because it contains a fraction under the radical.

The Third Guideline for Simple Radical Form. Don't leave radicals in the denominator of a fraction.

x	x^2
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100
11	121
12	144
13	169
14	196
15	225
16	256
17	289
18	324
19	361
20	400
21	441
22	484
23	529
24	576
25	625

Table 1. Squares.

Thus, David's $1/(2\sqrt{3})$ is not in simple radical form, because the denominator of his fraction contains a radical.

Only the equivalent form $\sqrt{3}/6$ obeys all three rules of simple radical form.

1. It is not possible to factor a perfect square from any radical in the expression $\sqrt{3}/6$.
2. There are no fractions under a radical in the expression $\sqrt{3}/6$.
3. The denominator in the expression $\sqrt{3}/6$ contains no radicals.

In this text and in this course, we will always follow the three guidelines for simple radical form.[8]

Simple Radical Form. When your answer is a radical expression:

1. If possible, factor out a perfect square.
2. Don't leave fractions under a radical.
3. Don't leave radicals in the denominator of a fraction.

In the examples that follow (and in the exercises), it is helpful if you know the squares of the first 25 positive integers. We've listed them in the margin for you in **Table 1** for future reference.

Let's place a few radical expressions in simple radical form. We'll start with some radical expressions that contain fractions under a radical.

► **Example 2.** *Place the expression* $\sqrt{1/8}$ *in simple radical form.*

The expression $\sqrt{1/8}$ contains a fraction under a radical. We could take the square root of both numerator and denominator, but that would produce $\sqrt{1}/\sqrt{8}$, which puts a radical in the denominator.

The better strategy is to change the form of 1/8 so that we have a perfect square in the denominator before taking the square root of the numerator and denominator. We note that if we multiply 8 by 2, the result is 16, a perfect square. This is hopeful, so we begin the simplification by multiplying both numerator and denominator of 1/8 by 2.

$$\sqrt{\frac{1}{8}} = \sqrt{\frac{1}{8} \cdot \frac{2}{2}} = \sqrt{\frac{2}{16}}$$

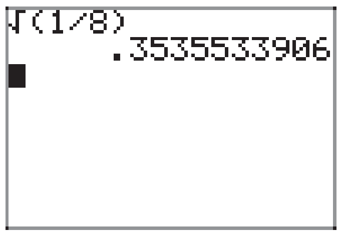
We now take the square root of both numerator and denominator. Because the denominator is now a perfect square, the result will not have a radical in the denominator.

$$\sqrt{\frac{2}{16}} = \frac{\sqrt{2}}{\sqrt{16}} = \frac{\sqrt{2}}{4}$$

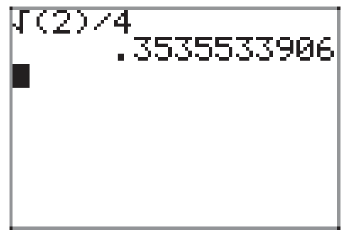
[8] In some courses, such as trigonometry and calculus, your instructor may relax these guidelines a bit. In some cases, it is easier to work with $1/\sqrt{2}$, for example, than it is to work with $\sqrt{2}/2$, even though they are equivalent.

This last result, $\sqrt{2}/4$ is in simple radical form. It is not possible to factor a perfect square from any radical, there are no fractions under any radical, and the denominator is free of radicals.

You can easily check your solution by using your calculator to compare the original expression with your simple radical form. In **Figure 4**(a), we've approximated the original expression, $\sqrt{1/8}$. In **Figure 4**(b), we've approximated our simple radical form, $\sqrt{2}/4$. Note that they yield identical decimal approximations.



(a) Approximating $\sqrt{1/8}$.



(b) Approximating $\sqrt{2}/4$.

Figure 4. Comparing $\sqrt{1/8}$ and $\sqrt{2}/4$.

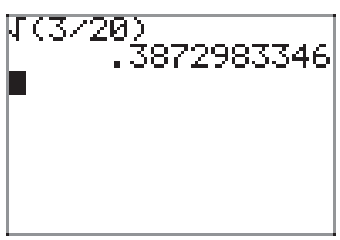
Let's look at another example.

► **Example 3.** *Place $\sqrt{3/20}$ in simple radical form.*

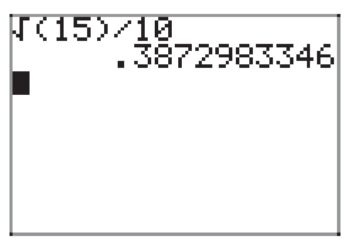
Following the lead from **Example 2**, we note that $5 \cdot 20 = 100$, a perfect square. So, we multiply both numerator and denominator by 5, then take the square root of both numerator and denominator once we have a perfect square in the denominator.

$$\sqrt{\frac{3}{20}} = \sqrt{\frac{3}{20} \cdot \frac{5}{5}} = \sqrt{\frac{15}{100}} = \frac{\sqrt{15}}{\sqrt{100}} = \frac{\sqrt{15}}{10}$$

Note that the decimal approximation of the simple radical form $\sqrt{15}/10$ in **Figure 5**(b) matches the decimal approximation of the original expression $\sqrt{3/20}$ in **Figure 5**(a).



(a) Approximating $\sqrt{3/20}$.



(b) Approximating $\sqrt{15}/10$.

Figure 5. Comparing the original $\sqrt{3/20}$ with the simple radical form $\sqrt{15}/10$.

We will now show how to deal with an expression having a radical in its denominator, but first we pause to explain a new piece of terminology.

Rationalizing the Denominator. The process of eliminating radicals from the denominator is called **rationalizing the denominator** because it results in a fraction where the denominator is free of radicals and is a rational number.

▶ **Example 4.** *Place the expression* $5/\sqrt{18}$ *in simple radical form.*

In the previous examples, making the denominator a perfect square seemed a good tactic. We apply the same tactic in this example, noting that $2 \cdot 18 = 36$ is a perfect square. However, the strategy is slightly different, as we begin the solution by multiplying both numerator and denominator by $\sqrt{2}$.

$$\frac{5}{\sqrt{18}} = \frac{5}{\sqrt{18}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

We now multiply numerators and denominators. In the denominator, the multiplication property of radicals is used, $\sqrt{18}\sqrt{2} = \sqrt{36}$.

$$\frac{5}{\sqrt{18}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{\sqrt{36}}$$

The strategy should now be clear. Because the denominator is a perfect square, $\sqrt{36} = 6$, clearing all radicals from the denominator of our result.

$$\frac{5\sqrt{2}}{\sqrt{36}} = \frac{5\sqrt{2}}{6}$$

The last result is in simple radical form. It is not possible to extract a perfect square root from any radical, there are no fractions under any radical, and the denominator is free of radicals.

In **Figure 6**, we compare the approximation for our original expression $5/\sqrt{18}$ with our simple radical form $5\sqrt{2}/6$.

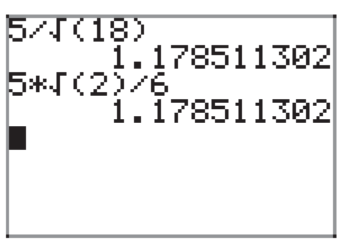


Figure 6. Comparing $5/\sqrt{18}$ with $5\sqrt{2}/6$.

Let's look at another example.

▶ **Example 5.** *Place the expression* $18/\sqrt{27}$ *in simple radical form.*

Note that $3 \cdot 27 = 81$ is a perfect square. We begin by multiplying both numerator and denominator of our expression by $\sqrt{3}$.

$$\frac{18}{\sqrt{27}} = \frac{18}{\sqrt{27}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

Multiply numerators and denominators. In the denominator, $\sqrt{27}\sqrt{3} = \sqrt{81}$.

$$\frac{18}{\sqrt{27}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{18\sqrt{3}}{\sqrt{81}}$$

Of course, $\sqrt{81} = 9$, so

$$\frac{18\sqrt{3}}{\sqrt{81}} = \frac{18\sqrt{3}}{9}$$

We can now reduce to lowest terms, dividing numerator and denominator by 9.

$$\frac{18\sqrt{3}}{9} = 2\sqrt{3}$$

In **Figure 7**, we compare approximations of the original expression $18/\sqrt{27}$ and its simple radical form $2\sqrt{3}$.

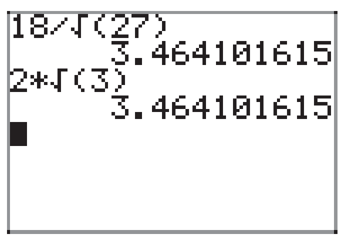


Figure 7. Comparing $18/\sqrt{27}$ with its simple radical form $2\sqrt{3}$.

Helpful Hints

In the previous section, we learned that if you square a product of exponential expressions, you multiply each of the exponents by 2.

$$(2^3 3^4 5^5)^2 = 2^6 3^8 5^{10}$$

Because taking the square root is the "inverse" of squaring,[9] we divide each of the exponents by 2.

[9] As we have pointed out in previous sections, taking the positive square root is the inverse of squaring, only if we restrict the domain of the squaring function to nonnegative real numbers, which we do here.

$$\sqrt{2^6 3^8 5^{10}} = 2^3 3^4 5^5$$

We also learned that prime factorization is an extremely powerful tool that is quite useful when placing radical expressions in simple radical form. We'll see that this is even more true in this section.

Let's look at an example.

▶ **Example 6.** *Place the expression* $\sqrt{1/98}$ *in simple radical form.*

Sometimes it is not easy to figure out how to scale the denominator to get a perfect square, even when provided with a table of perfect squares. This is when prime factorization can come to the rescue and provide a hint. So, first express the denominator as a product of primes in exponential form: $98 = 2 \cdot 49 = 2 \cdot 7^2$.

$$\sqrt{\frac{1}{98}} = \sqrt{\frac{1}{2 \cdot 7^2}}$$

We can now easily see what is preventing the denominator from being a perfect square. The problem is the fact that not all of the exponents in the denominator are divisible by 2. We can remedy this by multiplying both numerator and denominator by 2.

$$\sqrt{\frac{1}{2 \cdot 7^2}} = \sqrt{\frac{1}{2 \cdot 7^2} \cdot \frac{2}{2}} = \sqrt{\frac{2}{2^2 7^2}}$$

Note that each prime in the denominator now has an exponent that is divisible by 2. We can now take the square root of both numerator and denominator.

$$\sqrt{\frac{2}{2^2 7^2}} = \frac{\sqrt{2}}{\sqrt{2^2 7^2}}$$

Take the square root of the denominator by dividing each exponent by 2.

$$\frac{\sqrt{2}}{\sqrt{2^2 7^2}} = \frac{\sqrt{2}}{2^1 \cdot 7^1}$$

Then, of course, $2 \cdot 7 = 14$.

$$\frac{\sqrt{2}}{2 \cdot 7} = \frac{\sqrt{2}}{14}$$

In **Figure 8**, note how the decimal approximations of the original expression $\sqrt{1/98}$ and its simple radical form $\sqrt{2}/14$ match, strong evidence that we've found the correct simple radical form. That is, we cannot take a perfect square out of any radical, there are no fractions under any radical, and the denominators are clear of all radicals.

Let's look at another example.

▶ **Example 7.** *Place the expression* $12/\sqrt{54}$ *in simple radical form.*

Prime factor the denominator: $54 = 2 \cdot 27 = 2 \cdot 3^3$.

$$\frac{12}{\sqrt{54}} = \frac{12}{\sqrt{2 \cdot 3^3}}$$

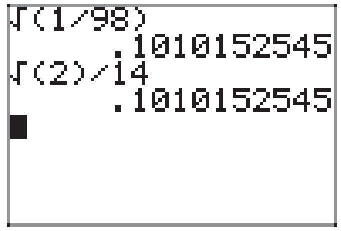


Figure 8. Comparing the original $\sqrt{1/98}$ with its simple radical form $\sqrt{2}/14$.

Neither prime in the denominator has an exponent divisible by 2. If we had another 2 and one more 3, then the exponents would be divisible by 2. This encourages us to multiply both numerator and denominator by $\sqrt{2 \cdot 3}$.

$$\frac{12}{\sqrt{2 \cdot 3^3}} = \frac{12}{\sqrt{2 \cdot 3^3}} \cdot \frac{\sqrt{2 \cdot 3}}{\sqrt{2 \cdot 3}} = \frac{12\sqrt{2 \cdot 3}}{\sqrt{2^2 3^4}}$$

Divide each of the exponents in the denominator by 2.

$$\frac{12\sqrt{2 \cdot 3}}{\sqrt{2^2 3^4}} = \frac{12\sqrt{2 \cdot 3}}{2^1 \cdot 3^2}$$

Then, in the numerator, $2 \cdot 3 = 6$, and in the denominator, $2 \cdot 3^2 = 18$.

$$\frac{12\sqrt{2 \cdot 3}}{2 \cdot 3^2} = \frac{12\sqrt{6}}{18}$$

Finally, reduce to lowest terms by dividing both numerator and denominator by 6.

$$\frac{12\sqrt{6}}{18} = \frac{2\sqrt{6}}{3}$$

In **Figure 9**, the approximation for the original expression $12/\sqrt{54}$ matches that of its simple radical form $2\sqrt{6}/3$.

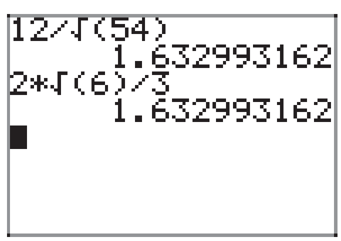


Figure 9. Comparing approximations of the original expression $12/\sqrt{54}$ with its simple radical form $2\sqrt{6}/3$.

Variable Expressions

If x is any real number, recall again that

$$\sqrt{x^2} = |x|.$$

If we combine the law of exponents for squaring a quotient with our property for taking the square root of a quotient, we can write

$$\sqrt{\left(\frac{a}{b}\right)^2} = \sqrt{\frac{a^2}{b^2}} = \frac{\sqrt{a^2}}{\sqrt{b^2}}$$

However, $\sqrt{(a/b)^2} = |a/b|$, while $\sqrt{a^2}/\sqrt{b^2} = |a|/|b|$. This discussion leads to the following key result.

Quotient Rule for Absolute Value. If a and b are any real numbers, then

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|},$$

provided $b \neq 0$. In words, the absolute value of a quotient is the quotient of the absolute values.

We saw this property previously in the chapter on the absolute value function, where we provided a different approach to the proof of the property. It's interesting that we can prove this property in a completely new way using the properties of square root. We'll see we have need for the Quotient Rule for Absolute Value in the examples that follow.

For example, if x is any real number except zero, using the quotient rule for absolute value we could write

$$\left|\frac{3}{x}\right| = \frac{|3|}{|x|} = \frac{3}{|x|}.$$

However, there is no way to remove the absolute value bars that surround x unless we know the sign of x. If $x > 0$ (remember, no zeros in the denominator), then $|x| = x$ and the expression becomes

$$\frac{3}{|x|} = \frac{3}{x}.$$

On the other hand, if $x < 0$, then $|x| = -x$ and the expression becomes

$$\frac{3}{|x|} = \frac{3}{-x} = -\frac{3}{x}.$$

Let's look at another example.

▶ **Example 8.** *Place the expression* $\sqrt{18/x^6}$ *in simple radical form. Discuss the domain.*

Note that x cannot equal zero, otherwise the denominator of $\sqrt{18/x^6}$ would be zero, which is not allowed. However, whether x is positive or negative, x^6 will be a positive number (raising a nonzero number to an even power always produces a positive real number), and $\sqrt{18/x^6}$ is well-defined.

Keeping in mind that x is nonzero, but could either be positive or negative, we proceed by first invoking **Property 1**, taking the positive square root of both numerator and denominator of our radical expression.

$$\sqrt{\frac{18}{x^6}} = \frac{\sqrt{18}}{\sqrt{x^6}}$$

From the numerator, we factor a perfect square. In the denominator, we use absolute value bars to insure a positive square root.

$$\frac{\sqrt{18}}{\sqrt{x^6}} = \frac{\sqrt{9}\sqrt{2}}{|x^3|} = \frac{3\sqrt{2}}{|x^3|}$$

We can use the Product Rule for Absolute Value to write $|x^3| = |x^2||x| = x^2|x|$. Note that we do not need to wrap x^2 in absolute value bars because x^2 is already positive.

$$\frac{3\sqrt{2}}{|x^3|} = \frac{3\sqrt{2}}{x^2|x|}$$

Because x could be positive or negative, we cannot remove the absolute value bars around x. We are done.

Let's look at another example.

▶ **Example 9.** *Place the expression* $\sqrt{12/x^5}$ *in simple radical form. Discuss the domain.*

Note that x cannot equal zero, otherwise the denominator of $\sqrt{12/x^5}$ would be zero, which is not allowed. Further, if x is a negative number, then x^5 will also be a negative number (raising a negative number to an odd power produces a negative number). If x were negative, then $12/x^5$ would also be negative and $\sqrt{12/x^5}$ would be undefined (you cannot take the square root of a negative number). Thus, x must be a positive real number or the expression $\sqrt{12/x^5}$ is undefined.

We proceed, keeping in mind that x is a positive real number. One possible approach is to first note that another factor of x is needed to make the denominator a perfect square. This motivates us to multiply both numerator and denominator inside the radical by x.

$$\sqrt{\frac{12}{x^5}} = \sqrt{\frac{12}{x^5} \cdot \frac{x}{x}} = \sqrt{\frac{12x}{x^6}}.$$

We can now use **Property 1** to take the square root of both numerator and denominator.

$$\sqrt{\frac{12x}{x^6}} = \frac{\sqrt{12x}}{\sqrt{x^6}}$$

In the numerator, we factor out a perfect square. In the denominator, absolute value bars would insure a positive square root. However, we've stated that x must be a positive number, so x^3 is already positive and absolute value bars are not needed.

$$\frac{\sqrt{12x}}{\sqrt{x^6}} = \frac{\sqrt{4}\sqrt{3x}}{x^3} = \frac{2\sqrt{3x}}{x^3}$$

Let's look at another example.

▶ **Example 10.** *Given that $x < 0$, place $\sqrt{27/x^{10}}$ in simple radical form.*

One possible approach would be to factor out a perfect square and write

$$\sqrt{\frac{27}{x^{10}}} = \sqrt{\frac{9}{x^{10}}}\sqrt{3} = \sqrt{\left(\frac{3}{x^5}\right)^2}\sqrt{3} = \left|\frac{3}{x^5}\right|\sqrt{3}.$$

Now, $|3/x^5| = |3|/(|x^4||x|) = 3/(x^4|x|)$, since $x^4 > 0$. Thus,

$$\left|\frac{3}{x^5}\right|\sqrt{3} = \frac{3}{x^4|x|}\sqrt{3}.$$

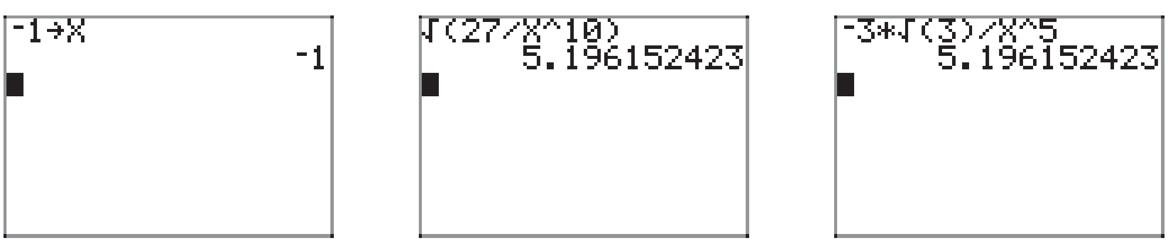
However, we are given that $x < 0$, so $|x| = -x$ and we can write

$$\frac{3}{x^4|x|}\sqrt{3} = \frac{3}{(x^4)(-x)}\sqrt{3} = -\frac{3}{x^5}\sqrt{3}.$$

We can move $\sqrt{3}$ into the numerator and write

$$-\frac{3}{x^5}\sqrt{3} = -\frac{3\sqrt{3}}{x^5}. \tag{11}$$

Again, it's instructive to test the validity of this result using your graphing calculator. Supposedly, the result is true for all values of $x < 0$. So, store -1 in x, then enter the original expression and its simple radical form, then compare the approximations, as shown in **Figures 10**(a), (b), and (c).



(a) Store -1 in x. (b) Approximate $\sqrt{27/x^{10}}$. (c) Approximate $-3\sqrt{3}/x^5$.

Figure 10. Comparing the original expression and its simple radical form at $x = -1$.

Alternative approach. A slightly different approach would again begin by taking the square root of both numerator and denominator.

$$\sqrt{\frac{27}{x^{10}}} = \frac{\sqrt{27}}{\sqrt{x^{10}}}$$

Now, $\sqrt{27} = \sqrt{9}\sqrt{3} = 3\sqrt{3}$ and we insure that $\sqrt{x^{10}}$ produces a positive number by using absolute value bars. That is, $\sqrt{x^{10}} = |x^5|$ and

$$\frac{\sqrt{27}}{\sqrt{x^{10}}} = \frac{3\sqrt{3}}{|x^5|}.$$

However, using the product rule for absolute value and the fact that $x^4 > 0$, $|x^5| = |x^4||x| = x^4|x|$ and

$$\frac{3\sqrt{3}}{|x^5|} = \frac{3\sqrt{3}}{x^4|x|}.$$

Finally, we are given that $x < 0$, so $|x| = -x$ and we can write

$$\frac{3\sqrt{3}}{x^4|x|} = \frac{3\sqrt{3}}{(x^4)(-x)} = -\frac{3\sqrt{3}}{x^5}. \tag{12}$$

Note that the simple radical form (**12**) of our alternative approach matches perfectly the simple radical form (**11**) of our first approach.

5.3 Exercises

1. Use a calculator to first approximate $\sqrt{5}/\sqrt{2}$. On the same screen, approximate $\sqrt{5/2}$. Report the results on your homework paper.

2. Use a calculator to first approximate $\sqrt{7}/\sqrt{5}$. On the same screen, approximate $\sqrt{7/5}$. Report the results on your homework paper.

3. Use a calculator to first approximate $\sqrt{12}/\sqrt{2}$. On the same screen, approximate $\sqrt{6}$. Report the results on your homework paper.

4. Use a calculator to first approximate $\sqrt{15}/\sqrt{5}$. On the same screen, approximate $\sqrt{3}$. Report the results on your homework paper.

In **Exercises 5-16**, place each radical expression in simple radical form. As in Example 2 in the narrative, check your result with your calculator.

5. $\sqrt{\dfrac{3}{8}}$

6. $\sqrt{\dfrac{5}{12}}$

7. $\sqrt{\dfrac{11}{20}}$

8. $\sqrt{\dfrac{3}{2}}$

9. $\sqrt{\dfrac{11}{18}}$

10. $\sqrt{\dfrac{7}{5}}$

11. $\sqrt{\dfrac{4}{3}}$

12. $\sqrt{\dfrac{16}{5}}$

13. $\sqrt{\dfrac{49}{12}}$

14. $\sqrt{\dfrac{81}{20}}$

15. $\sqrt{\dfrac{100}{7}}$

16. $\sqrt{\dfrac{36}{5}}$

In **Exercises 17-28**, place each radical expression in simple radical form. As in Example 4 in the narrative, check your result with your calculator.

17. $\dfrac{1}{\sqrt{12}}$

18. $\dfrac{1}{\sqrt{8}}$

19. $\dfrac{1}{\sqrt{20}}$

20. $\dfrac{1}{\sqrt{27}}$

21. $\dfrac{6}{\sqrt{8}}$

22. $\dfrac{4}{\sqrt{12}}$

23. $\dfrac{5}{\sqrt{20}}$

24. $\dfrac{9}{\sqrt{27}}$

25. $\dfrac{6}{2\sqrt{3}}$

26. $\dfrac{10}{3\sqrt{5}}$

27. $\dfrac{15}{2\sqrt{20}}$

28. $\dfrac{3}{2\sqrt{18}}$

In **Exercises 29-36**, place the given radical expression in simple form. Use prime factorization as in Example 8 in the narrative to help you with the calculations. As in Example 6, check your result with your calculator.

29. $\dfrac{1}{\sqrt{96}}$

30. $\dfrac{1}{\sqrt{432}}$

31. $\dfrac{1}{\sqrt{250}}$

32. $\dfrac{1}{\sqrt{108}}$

33. $\sqrt{\dfrac{5}{96}}$

34. $\sqrt{\dfrac{2}{135}}$

35. $\sqrt{\dfrac{2}{1485}}$

36. $\sqrt{\dfrac{3}{280}}$

In **Exercises 37-44**, place each of the given radical expressions in simple radical form. Make no assumptions about the sign of any variable. Variables can represent either positive or negative numbers.

37. $\sqrt{\dfrac{8}{x^4}}$

38. $\sqrt{\dfrac{12}{x^6}}$

39. $\sqrt{\dfrac{20}{x^2}}$

40. $\sqrt{\dfrac{32}{x^{14}}}$

41. $\dfrac{2}{\sqrt{8x^8}}$

42. $\dfrac{3}{\sqrt{12x^6}}$

43. $\dfrac{10}{\sqrt{20x^{10}}}$

44. $\dfrac{12}{\sqrt{6x^4}}$

In **Exercises 45-48**, follow the lead of Example 8 in the narrative to craft a solution.

45. Given that $x < 0$, place the radical expression $6/\sqrt{2x^6}$ in simple radical form. Check your solution on your calculator for $x = -1$.

46. Given that $x > 0$, place the radical expression $4/\sqrt{12x^3}$ in simple radical form. Check your solution on your calculator for $x = 1$.

47. Given that $x > 0$, place the radical expression $8/\sqrt{8x^5}$ in simple radical form. Check your solution on your calculator for $x = 1$.

48. Given that $x < 0$, place the radical expression $15/\sqrt{20x^6}$ in simple radical form. Check your solution on your calculator for $x = -1$.

In **Exercises 49-56**, place each of the radical expressions in simple form. Assume that all variables represent positive numbers.

49. $\sqrt{\dfrac{12}{x}}$

50. $\sqrt{\dfrac{18}{x}}$

51. $\sqrt{\dfrac{50}{x^3}}$

52. $\sqrt{\dfrac{72}{x^5}}$

53. $\dfrac{1}{\sqrt{50x}}$

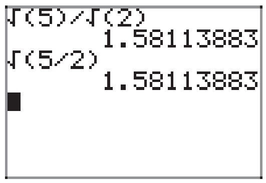
54. $\dfrac{2}{\sqrt{18x}}$

55. $\dfrac{3}{\sqrt{27x^3}}$

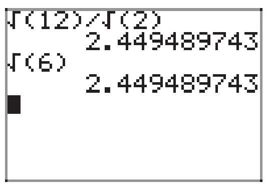
56. $\dfrac{5}{\sqrt{10x^5}}$

5.3 *Answers*

1.



3.



5. $\sqrt{6}/4$

7. $\sqrt{55}/10$

9. $\sqrt{22}/6$

11. $2\sqrt{3}/3$

13. $7\sqrt{3}/6$

15. $10\sqrt{7}/7$

17. $\sqrt{3}/6$

19. $\sqrt{5}/10$

21. $3\sqrt{2}/2$

23. $\sqrt{5}/2$

25. $\sqrt{3}$

27. $3\sqrt{5}/4$

29. $\sqrt{6}/24$

31. $\sqrt{10}/50$

33. $\sqrt{30}/24$

35. $\sqrt{330}/495$

37. $2\sqrt{2}/x^2$

39. $2\sqrt{5}/|x|$

41. $\sqrt{2}/(2x^4)$

43. $\sqrt{5}/(x^4|x|)$

45. $-3\sqrt{2}/x^3$

47. $2\sqrt{2x}/x^3$

49. $2\sqrt{3x}/x$

51. $5\sqrt{2x}/x^2$

53. $\sqrt{2x}/(10x)$

55. $\sqrt{3x}/(3x^2)$

5.4 Radical Expressions

In the previous two sections, we learned how to multiply and divide square roots. Specifically, we are now armed with the following two properties.

Property 1. Let a and b be any two real nonnegative numbers. Then,

$$\sqrt{a}\sqrt{b} = \sqrt{ab},$$

and, provided $b \neq 0$,

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}.$$

In this section, we will simplify a number of more extensive expressions containing square roots, particularly those that are fundamental to your work in future mathematics courses.

Let's begin by building some fundamental skills.

The Associative Property

We recall the associative property of multiplication.

Associative Property of Multiplication. Let a, b, and c be any real numbers. The *associative property of multiplication* states that

$$(ab)c = a(bc). \tag{2}$$

Note that the order of the numbers on each side of **equation (2)** has not changed. The numbers on each side of the equation are in the order a, b, and then c.

However, the grouping has changed. On the left, the parentheses around the product of a and b instruct us to perform that product first, then multiply the result by c. On the right, the grouping is different; the parentheses around b and c instruct us to perform that product first, then multiply by a. The key point to understand is the fact that the different groupings make no difference. We get the same answer in either case.

For example, consider the product $2 \cdot 3 \cdot 4$. If we multiply 2 and 3 first, then multiply the result by 4, we get

$$(2 \cdot 3) \cdot 4 = 6 \cdot 4 = 24.$$

On the other hand, if we multiply 3 and 4 first, then multiply the result by 2, we get

$$2 \cdot (3 \cdot 4) = 2 \cdot 12 = 24.$$

Note that we get the same result in either case. That is,

$$(2 \cdot 3) \cdot 4 = 2 \cdot (3 \cdot 4).$$

The associative property, seemingly trivial, takes on an extra level of sophistication if we apply it to expressions containing radicals. Let's look at an example.

► **Example 3.** *Simplify the expression* $3(2\sqrt{5})$. *Place your answer in simple radical form.*

Currently, the parentheses around 2 and $\sqrt{5}$ require that we multiply those two numbers first. However, the associative property of multiplication allows us to regroup, placing the parentheses around 3 and 2, multiplying those two numbers first, then multiplying the result by $\sqrt{5}$. We arrange the work as follows.

$$3(2\sqrt{5}) = (3 \cdot 2)\sqrt{5} = 6\sqrt{5}.$$

Readers should note the similarity to a very familiar manipulation.

$$3(2x) = (3 \cdot 2)x = 6x$$

In practice, when we became confident with this regrouping, we began to skip the intermediate step and simply state that $3(2x) = 6x$. In a similar vein, once you become confident with regrouping, you should simply state that $3(2\sqrt{5}) = 6\sqrt{5}$. If called upon to explain your answer, you must be ready to explain how you regrouped according to the associative property of multiplication. Similarly,

$$-4(5\sqrt{7}) = -20\sqrt{7}, \quad 12(5\sqrt{11}) = 60\sqrt{11}, \quad \text{and} \quad -5(-3\sqrt{3}) = 15\sqrt{3}.$$

The Commutative Property of Multiplication

We recall the commutative property of multiplication.

Commutative Property of Multiplication. Let a and b be any real numbers. The *commutative property of multiplication* states that

$$ab = ba. \tag{4}$$

The commutative property states that the order of multiplication is irrelevant. For example, $2 \cdot 3$ is the same as $3 \cdot 2$; they both equal 6. This seemingly trivial property, coupled with the associative property of multiplication, allows us to change the order of multiplication and regroup as we please.

▶ **Example 5.** *Simplify the expression* $\sqrt{5}(2\sqrt{3})$. *Place your answer in simple radical form.*

What we'd really like to do is first multiply $\sqrt{5}$ and $\sqrt{3}$. In order to do this, we must first regroup, then switch the order of multiplication as follows.

$$\sqrt{5}(2\sqrt{3}) = (\sqrt{5} \cdot 2)\sqrt{3} = (2\sqrt{5})\sqrt{3}$$

This is allowed by the associative and commutative properties of multiplication. Now, we regroup again and multiply.

$$(2\sqrt{5})\sqrt{3} = 2(\sqrt{5}\sqrt{3}) = 2\sqrt{15}$$

In practice, this is far too much work for such a simple calculation. Once we understand the associative and commutative properties of multiplication, the expression $a \cdot b \cdot c$ is unambiguous. Parentheses are not needed. We know that we can change the order of multiplication and regroup as we please. Therefore, when presented with the product of three numbers, simply multiply two of your choice together, then multiply the result by the third remaining number.

In the case of $\sqrt{5}(2\sqrt{3})$, we choose to first multiply $\sqrt{5}$ and $\sqrt{3}$, which is $\sqrt{15}$, then multiply this result by 2 to get $2\sqrt{15}$. Similarly,

$$\sqrt{5}(2\sqrt{7}) = 2\sqrt{35} \qquad \text{and} \qquad \sqrt{x}(3\sqrt{5}) = 3\sqrt{5x}.$$

▶ **Example 6.** *Simplify the expression* $\sqrt{6}(4\sqrt{8})$. *Place your answer in simple radical form.*

We start by multiplying $\sqrt{6}$ and $\sqrt{8}$, then the result by 4.

$$\sqrt{6}(4\sqrt{8}) = 4\sqrt{48}$$

Now, $48 = 16 \cdot 3$, so we can extract a perfect square.

$$4\sqrt{48} = 4(\sqrt{16}\sqrt{3}) = 4(4\sqrt{3})$$

Again, we choose to multiply the fours, then the result by the square root of three. That is,

$$4(4\sqrt{3}) = 16\sqrt{3}.$$

By induction, we can argue that the associative and commutative properties will allow us to group and arrange the product of more than three numbers in any order that we please.

▶ **Example 7.** *Simplify the expression* $(2\sqrt{12})(3\sqrt{3})$. *Place your answer in simple radical form.*

We'll first take the product of 2 and 3, then the product of $\sqrt{12}$ and $\sqrt{3}$, then multiply these results together.

$$(2\sqrt{12})(3\sqrt{3}) = (2 \cdot 3)(\sqrt{12}\sqrt{3}) = 6\sqrt{36}$$

Of course, $\sqrt{36} = 6$, so we can simplify further.

$$6\sqrt{36} = 6 \cdot 6 = 36$$

The Distributive Property

Recall the distributive property for real numbers.

Distributive Property. Let a, b, and c be any real numbers. Then,

$$a(b + c) = ab + ac. \tag{8}$$

You might recall the following operation, where you "distribute the 2," multiplying each term in the parentheses by 2.

$$2(3 + x) = 6 + 2x$$

You can do precisely the same thing with radical expressions.

$$2(3 + \sqrt{5}) = 6 + 2\sqrt{5}$$

Like the familiar example above, we "distributed the 2," multiplying each term in the parentheses by 2.

Let's look at more examples.

▶ **Example 9.** *Use the distributive property to expand the expression* $\sqrt{12}(3+\sqrt{3})$, *placing your final answer in simple radical form.*

First, distribute the $\sqrt{12}$, multiplying each term in the parentheses by $\sqrt{12}$. Note that $\sqrt{12}\sqrt{3} = \sqrt{36}$.

$$\sqrt{12}(3 + \sqrt{3}) = 3\sqrt{12} + \sqrt{36} = 3\sqrt{12} + 6$$

However, this last expression is not in simple radical form, as we can factor out a perfect square ($12 = 4 \cdot 3$).

$$\begin{aligned} 3\sqrt{12} + 6 &= 3(\sqrt{4}\sqrt{3}) + 6 \\ &= 3(2\sqrt{3}) + 6 \\ &= 6\sqrt{3} + 6 \end{aligned}$$

It doesn't matter whether the monomial factor is in the front or rear of the sum, you still distribute the monomial times each term in the parentheses.

► **Example 10.** *Use the distributive property to expand* $(\sqrt{3}+2\sqrt{2})\sqrt{6}$*. Place your answer in simple radical form.*

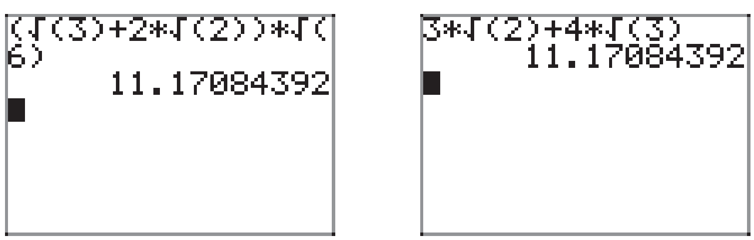
First, multiply each term in the parentheses by $\sqrt{6}$.

$$(\sqrt{3} + 2\sqrt{2})\sqrt{6} = \sqrt{18} + 2\sqrt{12}$$

To obtain the second term of this result, we chose to first multiply $\sqrt{2}$ and $\sqrt{6}$, which is $\sqrt{12}$, then we multiplied this result by 2. Now, we can factor perfect squares from both 18 and 12.

$$\begin{aligned} \sqrt{18} + 2\sqrt{12} &= \sqrt{9}\sqrt{2} + 2(\sqrt{4}\sqrt{3}) \\ &= 3\sqrt{2} + 2(2\sqrt{3}) \\ &= 3\sqrt{2} + 4\sqrt{3} \end{aligned}$$

Remember, you can check your results with your calculator. In **Figure 1**(a), we've found a decimal approximation for the original expression $(\sqrt{3} + 2\sqrt{2})\sqrt{6}$, and in **Figure 1**(b) we have a decimal approximation for our solution $3\sqrt{2} + 4\sqrt{3}$. Note that they are the same, providing evidence that our solution is correct.



(a) Approximating $(\sqrt{3} + 2\sqrt{2})\sqrt{6}$.

(b) Approximating $3\sqrt{2} + 4\sqrt{3}$.

Figure 1. Comparing the original expression with its simple radical form.

The distributive property is also responsible in helping us combine "like terms." For example, you might remember that $3x + 5x = 8x$, a seemingly simple calculation, but it is the distributive property that actually provides this solution. Note how we use the distributive property to factor x from each term.

$$3x + 5x = (3+5)x$$

Hence, $3x + 5x = 8x$. You can do the same thing with radical expressions.

$$3\sqrt{2} + 5\sqrt{2} = (3+5)\sqrt{2}$$

Hence, $3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$, and the structure of this result is identical to that shown in $3x + 5x = 8x$. There is no difference in the way we combine these "like terms." We repeat the common factor and add coefficients. For example,

$$2\sqrt{3} + 9\sqrt{3} = 11\sqrt{3}, \quad -4\sqrt{2} + 2\sqrt{2} = -2\sqrt{2}, \quad \text{and} \quad -3x\sqrt{x} + 5x\sqrt{x} = 2x\sqrt{x}.$$

In each case above, we're adding "like terms," by repeating the common factor and adding coefficients.

In the case that we don't have like terms, as in $3x + 5y$, there is nothing to be done. In like manner, each of the following expressions have no like terms that you can combine. They are as simplified as they are going to get.

$$3\sqrt{2} + 5\sqrt{3}, \quad 2\sqrt{11} - 8\sqrt{10}, \quad \text{and} \quad 2\sqrt{x} + 5\sqrt{y}$$

However, there are times when it can look as if you don't have like terms, but when you place everything in simple radical form, you discover that you do have like terms that can be combined by adding coefficients.

► **Example 11.** *Simplify the expression $5\sqrt{27} + 8\sqrt{3}$, placing the final expression in simple radical form.*

We can extract a perfect square $(27 = 9 \cdot 3)$.

$$\begin{aligned} 5\sqrt{27} + 8\sqrt{3} &= 5(\sqrt{9}\sqrt{3}) + 8\sqrt{3} \\ &= 5(3\sqrt{3}) + 8\sqrt{3} \\ &= 15\sqrt{3} + 8\sqrt{3} \end{aligned}$$

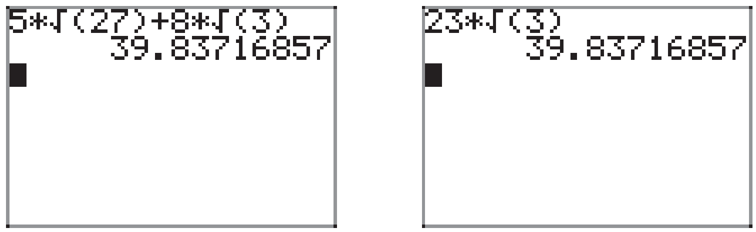
Note that we now have "like terms" that can be combined by adding coefficients.

$$15\sqrt{3} + 8\sqrt{3} = 23\sqrt{3}$$

A comparison of the original expression and its simplified form is shown in **Figures 2**(a) and (b).

► **Example 12.** *Simplify the expression $2\sqrt{20} + \sqrt{8} + 3\sqrt{5} + 4\sqrt{2}$, placing the result in simple radical form.*

We can extract perfect squares $(20 = 4 \cdot 5$ and $8 = 4 \cdot 2)$.



(a) Approximating $5\sqrt{27} + 8\sqrt{3}$.

(b) Approximating $23\sqrt{3}$.

Figure 2. Comparing the original expression with its simplified form.

$$\begin{aligned} 2\sqrt{20} + \sqrt{8} + 3\sqrt{5} + 4\sqrt{2} &= 2(\sqrt{4}\sqrt{5}) + \sqrt{4}\sqrt{2} + 3\sqrt{5} + 4\sqrt{2} \\ &= 2(2\sqrt{5}) + 2\sqrt{2} + 3\sqrt{5} + 4\sqrt{2} \\ &= 4\sqrt{5} + 2\sqrt{2} + 3\sqrt{5} + 4\sqrt{2} \end{aligned}$$

Now we can combine like terms by adding coefficients.

$$4\sqrt{5} + 2\sqrt{2} + 3\sqrt{5} + 4\sqrt{2} = 7\sqrt{5} + 6\sqrt{2}$$

Fractions can be a little tricky.

▶ **Example 13.** *Simplify $\sqrt{27} + 1/\sqrt{12}$, placing the result in simple radical form.*

We can extract a perfect square root $(27 = 9 \cdot 3)$. The denominator in the second term is $12 = 2^2 \cdot 3$, so one more 3 is needed in the denominator to make a perfect square.

$$\begin{aligned} \sqrt{27} + \frac{1}{\sqrt{12}} &= \sqrt{9}\sqrt{3} + \frac{1}{\sqrt{12}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \\ &= 3\sqrt{3} + \frac{\sqrt{3}}{\sqrt{36}} \\ &= 3\sqrt{3} + \frac{\sqrt{3}}{6} \end{aligned}$$

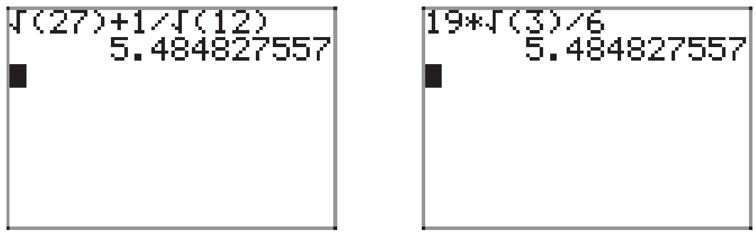
To add these fractions, we need a common denominator of 6.

$$\begin{aligned} 3\sqrt{3} + \frac{\sqrt{3}}{6} &= \frac{3\sqrt{3}}{1} \cdot \frac{6}{6} + \frac{\sqrt{3}}{6} \\ &= \frac{18\sqrt{3}}{6} + \frac{\sqrt{3}}{6} \end{aligned}$$

We can now combine numerators by adding coefficients.

$$\frac{18\sqrt{3}}{6} + \frac{\sqrt{3}}{6} = \frac{19\sqrt{3}}{6}$$

Decimal approximations of the original expression and its simplified form are shown in **Figures 3**(a) and (b).



(a) Approximating $\sqrt{27} + 1/\sqrt{12}$.

(b) Approximating $19\sqrt{3}/6$.

Figure 3. Comparing the original expression and its simple radical form.

At first glance, the lack of a monomial in the product $(x+1)(x+3)$ makes one think that the distributive property will not help us find the product. However, if we think of the second factor as a single unit, we can distribute it times each term in the first factor.

$$(x+1)(x+3) = x(x+3) + 1(x+3)$$

Apply the distributive property a second time, then combine like terms.

$$\begin{aligned} x(x+3) + 1(x+3) &= x^2 + 3x + x + 3 \\ &= x^2 + 4x + 3 \end{aligned}$$

We can handle products with radical expressions in the same manner.

► **Example 14.** *Simplify* $(2+\sqrt{2})(3+5\sqrt{2})$. *Place your result in simple radical form.*

Think of the second factor as a single unit and distribute it times each term in the first factor.

$$(2+\sqrt{2})(3+5\sqrt{2}) = 2(3+5\sqrt{2}) + \sqrt{2}(3+5\sqrt{2})$$

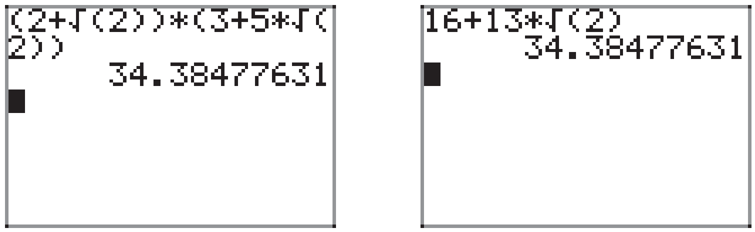
Now, use the distributive property again.

$$2(3+5\sqrt{2}) + \sqrt{2}(3+5\sqrt{2}) = 6 + 10\sqrt{2} + 3\sqrt{2} + 5\sqrt{4}$$

Note that in finding the last term, $\sqrt{2}\sqrt{2} = \sqrt{4}$. Now, $\sqrt{4} = 2$, then we can combine like terms.

$$\begin{aligned} 6 + 10\sqrt{2} + 3\sqrt{2} + 5\sqrt{4} &= 6 + 10\sqrt{2} + 3\sqrt{2} + 5(2) \\ &= 6 + 10\sqrt{2} + 3\sqrt{2} + 10 \\ &= 16 + 13\sqrt{2} \end{aligned}$$

Decimal approximations of the original expression and its simple radical form are shown in **Figures 4**(a) and (b).



(a) Approximating $(2 + \sqrt{2})(3 + 5\sqrt{2})$.

(b) Approximating $16 + 13\sqrt{2}$.

Figure 4. Comparing the original expression with its simple radical form.

Special Products

There are two special products that have important applications involving radical expressions, perhaps one more than the other. The first is the well-known difference of two squares pattern.

Difference of Squares. Let a and b be any numbers. Then,

$$(a + b)(a - b) = a^2 - b^2.$$

This pattern involves two binomial factors having identical first and second terms, the terms in one factor separated by a plus sign, the terms in the other factor separated by a minus sign. When we see this pattern of multiplication, we should square the first term of either factor, square the second term, then subtract the results. For example,

$$(2x + 3)(2x - 3) = 4x^2 - 9.$$

This special product applies equally well when the first and/or second terms involve radical expressions.

▶ **Example 15.** *Use the difference of squares pattern to multiply* $(2+\sqrt{11})(2-\sqrt{11})$.

Note that this multiplication has the form $(a+b)(a-b)$, so we apply the difference of squares pattern to get

$$(2 + \sqrt{11})(2 - \sqrt{11}) = (2)^2 - (\sqrt{11})^2.$$

Of course, $2^2 = 4$ and $(\sqrt{11})^2 = 11$, so we can finish as follows.

$$(2)^2 - (\sqrt{11})^2 = 4 - 11 = -7$$

► **Example 16.** *Use the difference of squares pattern to multiply* $(2\sqrt{5}+3\sqrt{7})(2\sqrt{5}-3\sqrt{7})$.

Again, this product has the special form $(a+b)(a-b)$, so we apply the difference of squares pattern to get

$$(2\sqrt{5}+3\sqrt{7})(2\sqrt{5}-3\sqrt{7}) = (2\sqrt{5})^2 - (3\sqrt{7})^2.$$

Next, we square a product of two factors according to the rule $(ab)^2 = a^2b^2$. Thus,

$$(2\sqrt{5})^2 = (2)^2(\sqrt{5})^2 = 4 \cdot 5 = 20$$

and

$$(3\sqrt{7})^2 = (3)^2(\sqrt{7})^2 = 9 \cdot 7 = 63.$$

Thus, we can complete the multiplication $(2\sqrt{5}+3\sqrt{7})(2\sqrt{5}-3\sqrt{7})$ with

$$(2\sqrt{5})^2 - (3\sqrt{7})^2 = 20 - 63 = -43.$$

This result is easily verified with a calculator, as shown in **Figure 5**.

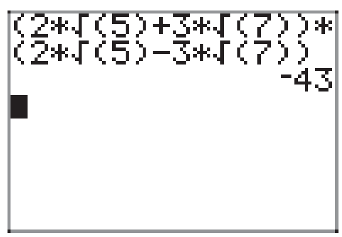


Figure 5. Approximating $(2\sqrt{5}+3\sqrt{7})(2\sqrt{5}-3\sqrt{7})$.

The second pattern of interest is the shortcut for squaring a binomial.

Squaring a Binomial. Let a and b be numbers. Then,

$$(a+b)^2 = a^2 + 2ab + b^2.$$

Here we square the first and second terms of the binomial, then produce the middle term of the result by multiplying the first and second terms and doubling that result. For example,

$$(2x+9)^2 = (2x)^2 + 2(2x)(9) + (9)^2 = 4x^2 + 36x + 81.$$

This pattern can also be applied to binomials containing radical expressions.

▶ **Example 17.** *Use the squaring a binomial pattern to expand* $(2\sqrt{x}+\sqrt{5})^2$*. Place your result in simple radical form. Assume that* x *is a positive real number* $(x > 0)$*.*

Applying the squaring a binomial pattern, we get

$$(2\sqrt{x}+\sqrt{5})^2 = (2\sqrt{x})^2 + 2(2\sqrt{x})(\sqrt{5}) + (\sqrt{5})^2.$$

As before, $(2\sqrt{x})^2 = (2)^2(\sqrt{x})^2 = 4x$ and $(\sqrt{5})^2 = 5$. In the case of $2(2\sqrt{x})(\sqrt{5})$, note that we are multiplying four numbers together. The associative and commutative properties state that we can multiply these four numbers in any order that we please. So, the product of 2 and 2 is 4, the product of $\sqrt{x}$ and $\sqrt{5}$ is $\sqrt{5x}$, then we multiply these results to produce the result $4\sqrt{5x}$. Thus,

$$(2\sqrt{x})^2 + 2(2\sqrt{x})(\sqrt{5}) + (\sqrt{5})^2 = 4x + 4\sqrt{5x} + 5.$$

Rationalizing Denominators

As we saw in the previous section, the instruction "rationalize the denominator" is a request to remove all radical expressions from the denominator. Of course, this is the "third guideline of simple radical form," but there are times, particularly in calculus, when the instruction changes to "rationalize the numerator." Of course, this is a request to remove all radicals from the numerator.

You can't have both worlds. You can either remove radical expressions from the denominator or from the numerator, but not both. If no instruction is given, assume that the "third guideline of simple radical form" is in play and remove all radical expressions from the denominator. We've already done a little of this in previous sections, but here we address a slightly more complicated type of expression.

▶ **Example 18.** *In the expression*

$$\frac{3}{2+\sqrt{2}},$$

rationalize the denominator.

The secret lies in the difference of squares pattern, $(a+b)(a-b) = a^2 - b^2$. For example,

$$(2+\sqrt{2})(2-\sqrt{2}) = (2)^2 - (\sqrt{2})^2 = 4 - 2 = 2.$$

This provides a terrific hint at how to proceed with rationalizing the denominator of the expression $3/(2+\sqrt{2})$. Multiply both numerator and denominator by $2-\sqrt{2}$.

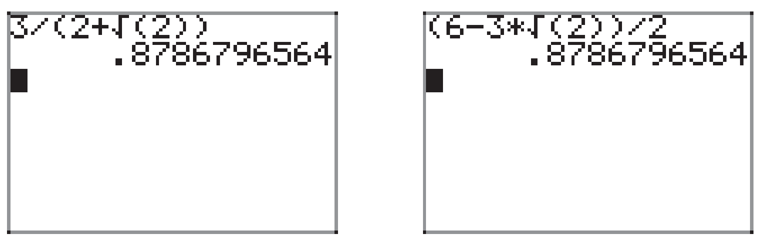
$$\frac{3}{2+\sqrt{2}} = \frac{3}{2+\sqrt{2}} \cdot \frac{2-\sqrt{2}}{2-\sqrt{2}}$$

Multiply numerators and denominators.

$$\begin{aligned}\frac{3}{2+\sqrt{2}} \cdot \frac{2-\sqrt{2}}{2-\sqrt{2}} &= \frac{3(2-\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})} \\ &= \frac{6-3\sqrt{2}}{(2)^2-(\sqrt{2})^2} \\ &= \frac{6-3\sqrt{2}}{4-2} \\ &= \frac{6-3\sqrt{2}}{2}\end{aligned}$$

Note that it is tempting to cancel the 2 in the denominator into the 6 in the numerator, but you are not allowed to cancel terms that are separated by a minus sign. This is a common error, so don't fall prey to this mistake.

In **Figures 6**(a) and (b), we compare decimal approximations of the original expression and its simple radical form.



(a) Approximating $3/(2+\sqrt{2})$.

(b) Approximating $(6-3\sqrt{2})/2$.

Figure 6. Comparing the original expression with its simple radical form.

▶ **Example 19.** *In the expression*

$$\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}},$$

rationalize the denominator.

Multiply numerator and denominator by $\sqrt{3}+\sqrt{2}$.

$$\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} \cdot \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}$$

Multiply numerators and denominators.

$$\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} \cdot \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \frac{(\sqrt{3}+\sqrt{2})^2}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})}$$

In the denominator, we have the difference of two squares. Thus,

$$(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2}) = (\sqrt{3})^2 - (\sqrt{2})^2 = 3 - 2 = 1.$$

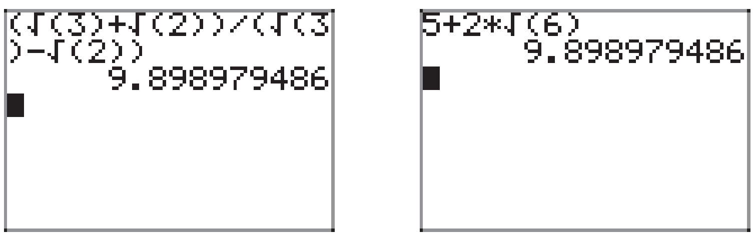
Note that this clears the denominator of radicals. This is the reason we multiply numerator and denominator by $\sqrt{3}+\sqrt{2}$. In the numerator, we can use the squaring a binomial shortcut to multiply.

$$\begin{aligned}(\sqrt{3}+\sqrt{2})^2 &= (\sqrt{3})^2 + 2(\sqrt{3})(\sqrt{2}) + (\sqrt{2})^2 \\ &= 3 + 2\sqrt{6} + 2 \\ &= 5 + 2\sqrt{6}\end{aligned}$$

Thus, we can complete the simplification started above.

$$\frac{(\sqrt{3}+\sqrt{2})^2}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})} = \frac{5+2\sqrt{6}}{1} = 5 + 2\sqrt{6}$$

In **Figures 7**(a) and (b), we compare the decimal approximations of the original expression with its simple radical form.



(a) Approximating $(\sqrt{3}+\sqrt{2})/(\sqrt{3}-\sqrt{2})$.

(b) Approximating $5+2\sqrt{6}$.

Figure 7. Comparing the original expression with its simple radical form.

Revisiting the Quadratic Formula

We can use what we've learned to place solutions provided by the quadratic formula in simple form. Let's look at an example.

► **Example 20.** *Solve the equation $x^2 = 2x + 2$ for x. Place your solution in simple radical form.*

The equation is nonlinear, so make one side zero.

$$x^2 - 2x - 2 = 0$$

Compare this result with the general form $ax^2 + bx + c = 0$ and note that $a = 1$, $b = -2$ and $c = -2$. Write down the quadratic formula, make the substitutions, then simplify.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)} = \frac{2 \pm \sqrt{12}}{2}$$

Note that we can factor a perfect square from the radical in the numerator.

$$x = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm \sqrt{4}\sqrt{3}}{2} = \frac{2 \pm 2\sqrt{3}}{2}$$

At this point, note that both numerator and denominator are divisible by 2. There are several ways that we can proceed with the reduction.

- Some people prefer to factor, then cancel.

$$x = \frac{2 \pm 2\sqrt{3}}{2} = \frac{2(1 \pm \sqrt{3})}{2} = \frac{\cancel{2}(1 \pm \sqrt{3})}{\cancel{2}} = 1 \pm \sqrt{3}$$

- Some prefer to use the distributive property.

$$x = \frac{2 \pm 2\sqrt{3}}{2} = \frac{2}{2} \pm \frac{2\sqrt{3}}{2} = 1 \pm \sqrt{3}$$

In each case, the final form of the answer is in simple radical form and it is reduced to lowest terms.

Warning 21. *When working with the quadratic formula, one of the most common algebra mistakes is to cancel addends instead of factors, as in*

$$\frac{2 \pm 2\sqrt{3}}{2} = \frac{\cancel{2} \pm 2\sqrt{3}}{\cancel{2}} = \pm 2\sqrt{3}.$$

Please try to avoid making this mistake.

Let's look at another example.

► **Example 22.** *Solve the equation* $3x^2 - 2x = 6$ *for* x*. Place your solution in simple radical form.*

This equation is nonlinear. Move every term to one side of the equation, making the other side of the equation equal to zero.

$$3x^2 - 2x - 6 = 0$$

Compare with the general form $ax^2 + bx + c = 0$ and note that $a = 3$, $b = -2$, and $c = -6$. Write down the quadratic formula and substitute.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-6)}}{2(3)} = \frac{2 \pm \sqrt{76}}{6}$$

Factor a perfect square from the radical in the numerator.

$$x = \frac{2 \pm \sqrt{76}}{6} = \frac{2 \pm \sqrt{4}\sqrt{19}}{6} = \frac{2 \pm 2\sqrt{19}}{6}$$

We choose to factor and cancel.

$$x = \frac{2 \pm 2\sqrt{19}}{6} = \frac{2(1 \pm \sqrt{19})}{2 \cdot 3} = \frac{2(1 \pm \sqrt{19})}{2 \cdot 3} = \frac{1 \pm \sqrt{19}}{3}$$

5.4 Exercises

In **Exercises 1-14**, place each of the radical expressions in simple radical form. Check your answer with your calculator.

1. $2(5\sqrt{7})$

2. $-3(2\sqrt{3})$

3. $-\sqrt{3}(2\sqrt{5})$

4. $\sqrt{2}(3\sqrt{7})$

5. $\sqrt{3}(5\sqrt{6})$

6. $\sqrt{2}(-3\sqrt{10})$

7. $(2\sqrt{5})(-3\sqrt{3})$

8. $(-5\sqrt{2})(-2\sqrt{7})$

9. $(-4\sqrt{3})(2\sqrt{6})$

10. $(2\sqrt{5})(-3\sqrt{10})$

11. $(2\sqrt{3})^2$

12. $(-3\sqrt{5})^2$

13. $(-5\sqrt{2})^2$

14. $(7\sqrt{11})^2$

In **Exercises 15-22**, use the distributive property to multiply. Place your final answer in simple radical form. Check your result with your calculator.

15. $2(3+\sqrt{5})$

16. $-3(4-\sqrt{7})$

17. $2(-5+4\sqrt{2})$

18. $-3(4-3\sqrt{2})$

19. $\sqrt{2}(2+\sqrt{2})$

20. $\sqrt{3}(4-\sqrt{6})$

21. $\sqrt{2}(\sqrt{10}+\sqrt{14})$

22. $\sqrt{3}(\sqrt{15}-\sqrt{33})$

In **Exercises 23-30**, combine like terms. Place your final answer in simple radical form. Check your solution with your calculator.

23. $-5\sqrt{2}+7\sqrt{2}$

24. $2\sqrt{3}+3\sqrt{3}$

25. $2\sqrt{6}-8\sqrt{6}$

26. $\sqrt{7}-3\sqrt{7}$

27. $2\sqrt{3}-4\sqrt{2}+3\sqrt{3}$

28. $7\sqrt{5}+2\sqrt{7}-3\sqrt{5}$

29. $2\sqrt{3}+5\sqrt{2}-7\sqrt{3}+2\sqrt{2}$

30. $3\sqrt{11}-2\sqrt{7}-2\sqrt{11}+4\sqrt{7}$

In **Exercises 31-40**, combine like terms where possible. Place your final answer in simple radical form. Use your calculator to check your result.

31. $\sqrt{45}+\sqrt{20}$

32. $-4\sqrt{45}-4\sqrt{20}$

33. $2\sqrt{18}-\sqrt{8}$

34. $-\sqrt{20}+4\sqrt{45}$

35. $-5\sqrt{27}+5\sqrt{12}$

36. $3\sqrt{12}-2\sqrt{27}$

37. $4\sqrt{20}+4\sqrt{45}$

38. $-2\sqrt{18}-5\sqrt{8}$

39. $2\sqrt{45}+5\sqrt{20}$

40. $3\sqrt{27}-4\sqrt{12}$

In **Exercises 41-48**, simplify each of the given rational expressions. Place your final answer in simple radical form. Check your result with your calculator.

41. $\sqrt{2}-\dfrac{1}{\sqrt{2}}$

42. $3\sqrt{3}-\dfrac{3}{\sqrt{3}}$

43. $2\sqrt{2}-\dfrac{2}{\sqrt{2}}$

44. $4\sqrt{5}-\dfrac{5}{\sqrt{5}}$

45. $5\sqrt{2}+\dfrac{3}{\sqrt{2}}$

46. $6\sqrt{3}+\dfrac{2}{\sqrt{3}}$

47. $\sqrt{8}-\dfrac{12}{\sqrt{2}}-3\sqrt{2}$

48. $\sqrt{27}-\dfrac{6}{\sqrt{3}}-5\sqrt{3}$

In **Exercises 49-60**, multiply to expand each of the given radical expressions. Place your final answer in simple radical form. Use your calculator to check your result.

49. $(2+\sqrt{3})(3-\sqrt{3})$

50. $(5+\sqrt{2})(2-\sqrt{2})$

51. $(4+3\sqrt{2})(2-5\sqrt{2})$

52. $(3+5\sqrt{3})(1-2\sqrt{3})$

53. $(2+3\sqrt{2})(2-3\sqrt{2})$

54. $(3+2\sqrt{5})(3-2\sqrt{5})$

55. $(2\sqrt{3}+3\sqrt{2})(2\sqrt{3}-3\sqrt{2})$

56. $(8\sqrt{2}+\sqrt{5})(8\sqrt{2}-\sqrt{5})$

57. $(2+\sqrt{5})^2$

58. $(3-\sqrt{2})^2$

59. $(\sqrt{3}-2\sqrt{5})^2$

60. $(2\sqrt{3}+3\sqrt{2})^2$

In **Exercises 61-68**, place each of the given rational expressions in simple radical form by "rationalizing the denominator." Check your result with your calculator.

61. $\dfrac{1}{\sqrt{5}+\sqrt{3}}$

62. $\dfrac{1}{2\sqrt{3}-\sqrt{2}}$

63. $\dfrac{6}{2\sqrt{5}-\sqrt{2}}$

64. $\dfrac{9}{3\sqrt{3}-\sqrt{6}}$

65. $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$

66. $\dfrac{3-\sqrt{5}}{3+\sqrt{5}}$

67. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

68. $\dfrac{2\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

In **Exercises 69-76**, use the quadratic formula to find the solutions of the given equation. Place your solutions in simple radical form and reduce your solutions to lowest terms.

69. $3x^2 - 8x = 5$

70. $5x^2 - 2x = 1$

71. $5x^2 = 2x + 1$

72. $3x^2 - 2x = 11$

73. $7x^2 = 6x + 2$

74. $11x^2 + 6x = 4$

75. $x^2 = 2x + 19$

76. $100x^2 = 40x - 1$

In **Exercises 77-80**, we will suspend the usual rule that you should rationalize the denominator. Instead, just this one time, rationalize the numerator of the resulting expression.

77. Given $f(x) = \sqrt{x}$, evaluate the expression

$$\frac{f(x) - f(2)}{x - 2},$$

and then "rationalize the numerator."

78. Given $f(x) = \sqrt{x+2}$, evaluate the expression

$$\frac{f(x) - f(3)}{x - 3},$$

and then "rationalize the numerator."

79. Given $f(x) = \sqrt{x}$, evaluate the expression

$$\frac{f(x+h) - f(x)}{h},$$

and then "rationalize the numerator."

80. Given $f(x) = \sqrt{x-3}$, evaluate the expression

$$\frac{f(x+h) - f(x)}{h},$$

and then "rationalize the numerator."

5.4 Answers

1. $10\sqrt{7}$

3. $-2\sqrt{15}$

5. $15\sqrt{2}$

7. $-6\sqrt{15}$

9. $-24\sqrt{2}$

11. 12

13. 50

15. $6+2\sqrt{5}$

17. $-10+8\sqrt{2}$

19. $2\sqrt{2}+2$

21. $2\sqrt{5}+2\sqrt{7}$

23. $2\sqrt{2}$

25. $-6\sqrt{6}$

27. $5\sqrt{3}-4\sqrt{2}$

29. $7\sqrt{2}-5\sqrt{3}$

31. $5\sqrt{5}$

33. $4\sqrt{2}$

35. $-5\sqrt{3}$

37. $20\sqrt{5}$

39. $16\sqrt{5}$

41. $\sqrt{2}/2$

43. $\sqrt{2}$

45. $13\sqrt{2}/2$

47. $-7\sqrt{2}$

49. $3+\sqrt{3}$

51. $-22-14\sqrt{2}$

53. -14

55. -6

57. $9+4\sqrt{5}$

59. $23-4\sqrt{15}$

61. $\dfrac{\sqrt{5}-\sqrt{3}}{2}$

63. $\dfrac{2\sqrt{5}+\sqrt{2}}{3}$

65. $7+4\sqrt{3}$

67. $5+2\sqrt{6}$

69. $(4\pm\sqrt{31})/3$

71. $(1\pm\sqrt{6})/5$

73. $(3\pm\sqrt{23})/7$

75. $1\pm2\sqrt{5}$

77. $\dfrac{1}{\sqrt{x}+\sqrt{2}}$

79. $\dfrac{1}{\sqrt{x+h}+\sqrt{x}}$

5.5 Radical Equations

In this section we are going to solve equations that contain one or more radical expressions. In the case where we can *isolate the radical expression on one side of the equation*, we can simply raise both sides of the equation to a power that will eliminate the radical expression. For example, if

$$\sqrt{x-1} = 2, \tag{1}$$

then we can square both sides of the equation, eliminating the radical.

$$\begin{aligned} \left(\sqrt{x-1}\right)^2 &= (2)^2 \\ x-1 &= 4 \end{aligned}$$

Now that the radical is eliminated, we can appeal to well understood techniques to solve the equation that remains. In this case, we need only add 1 to both sides of the equation to obtain

$$x = 5.$$

This solution is easily checked. Substitute $x = 5$ in the original **equation (1)**.

$$\begin{aligned} \sqrt{x-1} &= 2 \\ \sqrt{5-1} &= 2 \\ \sqrt{4} &= 2 \end{aligned}$$

The last line is valid because the "positive square root of 4" is indeed 2.

This seems pretty straight forward, but there are some subtleties. Let's look at another example, one with an equation quite similar to **equation (1)**.

► **Example 2.** *Solve the equation* $\sqrt{x-1} = -2$ *for* x.

If you carefully study the equation

$$\sqrt{x-1} = -2, \tag{3}$$

you might immediately detect a difficulty. The left-hand side of the equation calls for a "positive square root," but the right-hand side of the equation is negative. Intuitively, there can be no solutions.

A look at the graphs of each side of the equation also reveals the problem. The graphs of $y = \sqrt{x-1}$ and $y = -2$ are shown in **Figure 1**. Note that the graphs do not intersect, so the equation $\sqrt{x-1} = -2$ has no solution.

However, note what happens when we square both sides of **equation (3)**.

$$\begin{aligned} (\sqrt{x-1})^2 &= (-2)^2 \\ x-1 &= 4 \end{aligned} \tag{4}$$

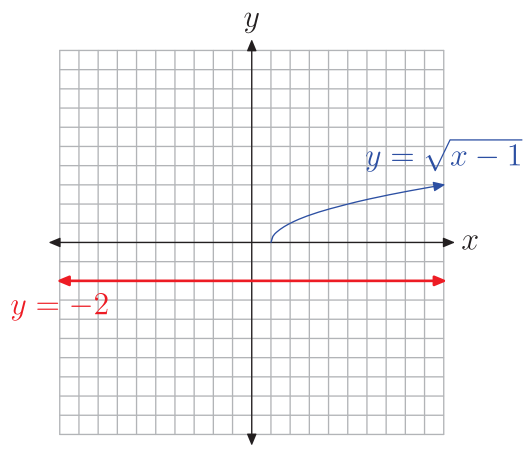


Figure 1. The graphs of $y = \sqrt{x-1}$ and $y = -2$ do not intersect.

This result is identical to the result we got when we squared both sides of the equation $\sqrt{x-1} = 2$ above. If we continue, adding 1 to both sides of the equation, we get

$$x = 5.$$

But this cannot be correct, as both intuition and the graphs in **Figure 1** have shown that the equation $\sqrt{x-1} = -2$ has no solutions.

Let's check the solution $x = 5$ in the original **equation (3)**.

$$\begin{aligned}\sqrt{x-1} &= -2\\ \sqrt{5-1} &= -2\\ \sqrt{4} &= -2\end{aligned}$$

Because the "positive square root of 4" does not equal −2, this last line is incorrect and the "solution" $x = 5$ does not check in the equation $\sqrt{x-1} = -2$. Because the only solution we found does not check, the equation has no solutions.

The discussion in **Example 2** dictates caution.

Warning 5. *Whenever you square both sides of an equation, there is a possibility that you can introduce* extraneous *solutions, "extra" solutions that will not check in the original problem.*

There is only one way to avoid this dilemma of extraneous equations.

Checking Solutions. Whenever you square both sides of an equation, you **must** check each of your solutions in the **original** equation. This is the only way you can be sure you have a valid solution.

Squaring a Binomial

As we've seen time and time again, the squaring a binomial pattern is of utmost importance.

Squaring a Binomial. If a and b are any real numbers, then

$$(a+b)^2 = a^2 + 2ab + b^2.$$

The squaring a binomial pattern will play a major role in the rest of the examples in this section.

Let's look at some examples of its use.

► **Example 6.** *Expand and simplify* $(1+\sqrt{x})^2$ *by using the squaring a binomial pattern. Assume that* $x \geq 0$.

The assumption that $x \geq 0$ is required, otherwise the expression $\sqrt{x}$ involves the square root of a negative number, which is not a real number.

The squaring a binomial pattern tells us to square the first and second terms. However, there is also a middle term, which is found by taking the product of the first and second terms, then multiplying the result by 2.

$$\begin{aligned} (1+\sqrt{x})^2 &= (1)^2 + 2(1)(\sqrt{x}) + (\sqrt{x})^2 \\ &= 1 + 2\sqrt{x} + x \end{aligned}$$

Let's look at another example.

► **Example 7.** *Expand and simplify* $(\sqrt{x+1}-\sqrt{x})^2$ *by using the squaring a binomial pattern. Comment on the domain of this expression.*

In order for this expression to make sense, we must avoid taking the square root of a negative number. Hence, both expressions under the square roots must be nonnegative (positive or zero). That is,

$$x+1 \geq 0 \qquad \text{and} \qquad x \geq 0$$

Solving each of these inequalities independently, we get the fact that

$$x \geq -1 \qquad \text{and} \qquad x \geq 0.$$

Because of the word "and," the requested domain is the set of all numbers that satisfy *both* inequalities, namely, the set of all real numbers that are greater than or equal to zero. That is, the domain of the expression is $\{x : x \geq 0\}$.

We will now expand the expression $(\sqrt{x+1}-\sqrt{x})^2$ using the squaring a binomial pattern.

$$
\begin{aligned}
(\sqrt{x+1}-\sqrt{x})^2 &= (\sqrt{x+1})^2 - 2(\sqrt{x+1})(\sqrt{x}) + (\sqrt{x})^2 \\
&= x+1+2\sqrt{(x+1)x}+x \\
&= 2x+1+2\sqrt{x^2+x}
\end{aligned}
$$

Isolate the Radical

Our mantra will be the strategy phrase "Isolate the radical."

> **Isolate the Radical.** When you solve equations containing one radical, isolate the radical by itself on one side of the equation.

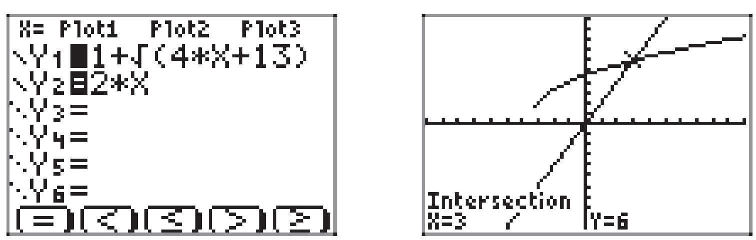
Although this is not always possible (some equations might contain more than one radical exprssion), it is possible in our next example.

▶ **Example 8.** *Solve the equation*

$$1+\sqrt{4x+13}=2x \tag{9}$$

for x.

Let's look at a graphing calculator solution. We've loaded the left- and right-hand sides of $1+\sqrt{4x+13}=2x$ into `Y1` and `Y2`, respectively, as shown in **Figure 2**(a). We then use `6:ZStandard` and the `intersect` utility on the `CALC` menu to determine the coordinates of the point of intersection of $y=1+\sqrt{4x+13}$ and $y=2x$, as shown in **Figure 2**(b).



(a) Loading $y=1+\sqrt{4x+13}$ and $y=2x$ into the `Y=` menu.

(b) The solution is $x\approx 3$.

Figure 2. Solving $1+\sqrt{4x+13}=2x$ on the graphing calculator. Note that there is only one point of intersection.

We will now present an algebraic solution, but note that we are forewarned that there is only one solution and we believe that the solution is $x\approx 3$. Of course, this is only an approximation, as is always the case when we pick up our calculator (our approximating machine).

Chant the strategy phrase "isolate the radical," then isolate the radical on one side of the equation. We will accomplish this directive by subtracting 1 from both sides of the equation.

$$\begin{aligned} 1 + \sqrt{4x + 13} &= 2x \\ \sqrt{4x + 13} &= 2x - 1 \end{aligned}$$

Next, square both sides of the equation.

$$(\sqrt{4x + 13})^2 = (2x - 1)^2$$

Squaring eliminates the radical on the left, but we must use the squaring a binomial pattern to square the binomial on the right-side of the equation.

$$\begin{aligned} 4x + 13 &= (2x)^2 - 2(2x)(1) + (1)^2 \\ 4x + 13 &= 4x^2 - 4x + 1 \end{aligned}$$

We've succeed in clearing all square roots from the equation with our "isolate the radical" strategy. The equation that remains is nonlinear (there is a power of x higher than 1), so we want to make one side of the equation equal to zero. We will do this by subtracting $4x$ and 13 from both sides of the equation.

$$\begin{aligned} 0 &= 4x^2 - 4x + 1 - 4x - 13 \\ 0 &= 4x^2 - 8x - 12 \end{aligned}$$

At this point, note that each term on the right-hand side of the equation is divisible by 4. Divide both sides of the equation by 4, then use the *ac*-test to factor the result.

$$\begin{aligned} 0 &= x^2 - 2x - 3 \\ 0 &= (x - 3)(x + 1) \end{aligned}$$

Set each factor on the right-hand side of this last equation to obtain the solutions $x = 3$ and $x = -1$.

Note that $x = 3$ matches the solution found by graphing in **Figure 2**(b). However, an "extra" solution $x = -1$ has appeared. Remember that we squared both sides of the original equation, so it is possible that extraneous solutions have been introduced. We need to check each of our solutions by substituting them into the **original** equation.

Our graph in **Figure 2**(b) adds credence to the analytical solution $x = 3$, so let's check that one first. Substitute $x = 3$ in the original equation.

$$\begin{aligned} 1 + \sqrt{4x + 13} &= 2x \\ 1 + \sqrt{4(3) + 13} &= 2(3) \\ 1 + \sqrt{25} &= 6 \\ 1 + 5 &= 6 \end{aligned}$$

Clearly, $x = 3$ checks and is a valid solution.

Next, let's check the "suspect" solution $x = -1$ by substituting it into the original equation.

$$\begin{aligned} 1 + \sqrt{4x+13} &= 2x \\ 1 + \sqrt{4(-1)+13} &= 2(-1) \\ 1 + \sqrt{9} &= -2 \\ 1 + 3 &= -2 \end{aligned}$$

Clearly, $x = -1$ does not check and is not a solution.

Thus, the only solution of $1 + \sqrt{4x+13} = 2x$ is $x = 3$. Readers should take note how that graphical solution and the analytic solution complement one another.

Before looking at another example, let's look at one of the most common mistakes made in the algebraic solution of **equation (9)**.

A Common Algebraic Mistake

In this section we discuss one of the most common algebraic mistakes encountered when solving equations that contain radical expressions.

Warning 10. *Many of the computations in this section are* **incorrect**. *They are examples of common algebra mistakes made when solving equations containing radicals. Keep this in mind and read the material in this section* **very** *carefully.*

When presented with the equation

$$1 + \sqrt{4x+13} = 2x, \tag{11}$$

some will square both sides of the equation in the following manner.

$$(1)^2 + (\sqrt{4x+13})^2 = (2x)^2, \tag{12}$$

arriving at

$$1 + 4x + 13 = 4x^2.$$

Make one side zero, then divide both sides of the resulting equation by 2.

$$\begin{aligned} 0 &= 4x^2 - 4x - 14 \\ 0 &= 2x^2 - 2x - 7 \end{aligned}$$

The careful reader will already realize that we've traveled the wrong path, as this result is quite different from that at a similar point in the solution of **Example 8**. However, we can continue with the solution by using the quadratic formula to solve the last equation for x. When we compare $2x^2 - 2x - 7$ with $ax^2 + bx + c$, note that $a = 2$, $b = -2$, and $c = -7$. Thus,

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-7)}}{2(2)} \\ &= \frac{2 \pm \sqrt{60}}{4}. \end{aligned}$$

However, neither of these "solutions" represent the correct solution found in **Example 8**, namely, $x = 3$. So, what have we done wrong?

The mistake occurred in the very first step when we squared both sides of the **equation (11)**. Indeed, to get **equation (12)**, we did not actually square both sides of **equation (11)**. Rather, we squared each of the individual terms on each side of the equation.

This is a serious mistake. In essence, we started with an equation having the form

$$a + b = c, \tag{13}$$

then squared "both sides" in the following manner.

$$a^2 + b^2 = c^2. \tag{14}$$

This is not valid. For example, start with

$$2 + 3 = 5,$$

a completely valid equation as the sum of 2 and 3 is 5. Now "square" as we did in **equation (14)** to get

$$2^2 + 3^2 = 5^2.$$

However, note that this simplifies as

$$4 + 9 = 25,$$

so we no longer have a valid equation.

The mistake made here is that we squared each of the individual terms on each side of the equation instead of squaring "each side" of the equation. If we had done that, we would have been all right, as is seen in this calculation.

$$\begin{aligned} 2 + 3 &= 5 \\ (2 + 3)^2 &= 5^2 \\ 2^2 + 2(2)(3) + 3^2 &= 5^2 \\ 4 + 12 + 9 &= 25 \end{aligned}$$

Just remember, $a+b=c$ does not imply $a^2+b^2=c^2$.

Warning 15. *We will now return to correct computations.*

More than One Radical

Let's look at an equation that contains more than one radical.

▶ **Example 16.** *Solve the equation*

$$\sqrt{2x}+\sqrt{2x+3}=3 \tag{17}$$

for x.

We'll start with a graphical solution of the equation. First, load the equations $y=\sqrt{2x}+\sqrt{2x+3}$ and $y=3$ into the `Y=` menu, as shown in **Figure 3**(a).

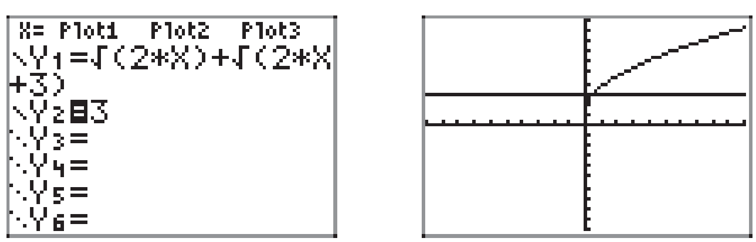
We cannot take the square root of a negative number, so when we consider the function defined by the equation $y=\sqrt{2x}+\sqrt{2x+3}$, both expressions under the radicals must be nonnegative. That is,

$$2x \geq 0 \qquad \text{and} \qquad 2x+3 \geq 0.$$

Solving each of these independently,

$$x \geq 0 \qquad \text{and} \qquad x \geq -\frac{3}{2}.$$

The numbers that are greater than or equal to zero *and* greater than or equal to $-3/2$ are the numbers greater than or equal to zero. Hence, the domain of the function defined by the equation $y=\sqrt{2x}+\sqrt{2x+3}$ is $\{x : x \geq 0\}$. Thus, it should not come as a shock when the graph of $y=\sqrt{2x}+\sqrt{2x+3}$ lies entirely to the right of zero, as shown in **Figure 3**(b).

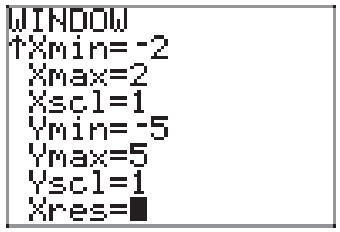


(a) Load each side of **equation (17)** into `Y1` and `Y2`.

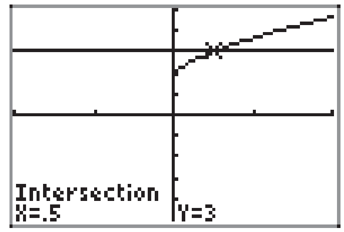
(b) The graph of $y=\sqrt{2x}+\sqrt{2x+3}$ lies completely to the right of zero.

Figure 3. Drawing the graphs of $y=\sqrt{2x}+\sqrt{2x+3}$ and $y=3$.

It's a bit difficult to see the point of intersection in **Figure 3**(b), so let's adjust the `WINDOW` settings as shown in **Figure 4**(a). As you can see **Figure 4**(b), this highlights the point of intersection a bit more clearly and the `5:intersect` utility in the `CALC` menu finds the point of intersection shown in **Figure 4**(b).



(a) Adjust the view.



(b) Use `5:intersect` to find the point of intersection.

Figure 4. Solving $\sqrt{2x} + \sqrt{2x+3} = 3$ graphically.

The graphing calculator reports one solution (there's only one point of intersection), and the x-value of the point of intersection is approximately $x \approx 0.5$.

Now, let's look at an algebraic solution. Since are two radical expressions in this equation, we will isolate one of them on one side of the equation. We choose to isolate the more complex of the two radical expressions on the left-hand side of the equation, then square both sides of the resulting equation.

$$\begin{aligned} \sqrt{2x} + \sqrt{2x+3} &= 3 \\ \sqrt{2x+3} &= 3 - \sqrt{2x} \\ (\sqrt{2x+3})^2 &= (3 - \sqrt{2x})^2 \end{aligned}$$

On the left, squaring eliminates the radical. To square the binomial on the right, we use the squaring a binomial pattern to obtain

$$\begin{aligned} 2x + 3 &= (3)^2 - 2(3)(\sqrt{2x}) + (\sqrt{2x})^2 \\ 2x + 3 &= 9 - 6\sqrt{2x} + 2x. \end{aligned}$$

We still have one radical expression left on the right-hand side of this equation, so we'll follow the mantra "isolate the radical." First, subtract $2x$ from both sides of the equation to obtain

$$3 = 9 - 6\sqrt{2x},$$

then subtract 9 from both sides of the equation.

$$-6 = -6\sqrt{2x}$$

We've succeeded in isolating the radical term on one side of the equation. Now, divide both sides of the equation by -6, then square both sides of the resulting equation.

$$\begin{aligned} 1 &= \sqrt{2x} \\ (1)^2 &= (\sqrt{2x})^2 \\ 1 &= 2x \end{aligned}$$

Divide both sides of the last result by 2.

$$x = \frac{1}{2}$$

Note that this agrees nicely with our graphical solution ($x \approx 0.5$), but let's check our solution by substituting $x = 1/2$ into the **original** equation.

$$\begin{aligned} \sqrt{2x} + \sqrt{2x+3} &= 3 \\ \sqrt{2(1/2)} + \sqrt{2(1/2)+3} &= 3 \\ \sqrt{1} + \sqrt{4} &= 3 \\ 1 + 2 &= 3 \end{aligned}$$

This last statement is true, so the solution $x = 1/2$ checks.

5.5 Exercises

For the rational functions in **Exercises 1-6**, perform each of the following tasks.

i. Load the function f and the line $y = k$ into your graphing calculator. Adjust the viewing window so that all point(s) of intersection of the two graphs are visible in your viewing window.
ii. Copy the image in your viewing window onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Label the graphs with their equations. *Remember to draw all lines with a ruler.*
iii. Use the `intersect` utility to determine the coordinates of the point(s) of intersection. Plot the point of intersection on your homework paper and label it with its coordinates.
iv. Solve the equation $f(x) = k$ algebraically. Place your work and solution next to your graph. Do the solutions agree?

1. $f(x) = \sqrt{x+3}$, $k = 2$

2. $f(x) = \sqrt{4-x}$, $k = 3$

3. $f(x) = \sqrt{7-2x}$, $k = 4$

4. $f(x) = \sqrt{3x+5}$, $k = 5$

5. $f(x) = \sqrt{5+x}$, $k = 4$

6. $f(x) = \sqrt{4-x}$, $k = 5$

In **Exercises 7-12**, use an algebraic technique to solve the given equation. Check your solutions.

7. $\sqrt{-5x+5} = 2$

8. $\sqrt{4x+6} = 7$

9. $\sqrt{6x-8} = 8$

10. $\sqrt{2x+4} = 2$

11. $\sqrt{-3x+1} = 3$

12. $\sqrt{4x+7} = 3$

For the rational functions in **Exercises 13-16**, perform each of the following tasks.

i. Load the function f and the line $y = k$ into your graphing calculator. Adjust the viewing window so that all point(s) of intersection of the two graphs are visible in your viewing window.
ii. Copy the image in your viewing window onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Label the graphs with their equations. *Remember to draw all lines with a ruler.*
iii. Use the `intersect` utility to determine the coordinates of the point(s) of intersection. Plot the point of intersection on your homework paper and label it with its coordinates.
iv. Solve the equation $f(x) = k$ algebraically. Place your work and solution next to your graph. Do the solutions agree?

13. $f(x) = \sqrt{x+3} + x$, $k = 9$

14. $f(x) = \sqrt{x+6} - x$, $k = 4$

15. $f(x) = \sqrt{x-5} - x$, $k = -7$

16. $f(x) = \sqrt{x+5} + x$, $k = 7$

In **Exercises 17-24**, use an algebraic technique to solve the given equation. Check your solutions.

17. $\sqrt{x+1}+x=5$

18. $\sqrt{x+8}-x=8$

19. $\sqrt{x+4}+x=8$

20. $\sqrt{x+8}-x=2$

21. $\sqrt{x+5}-x=3$

22. $\sqrt{x+5}+x=7$

23. $\sqrt{x+9}-x=9$

24. $\sqrt{x+7}+x=5$

For the rational functions in **Exercises 25-28**, perform each of the following tasks.

i. Load the function f and the line $y=k$ into your graphing calculator. Adjust the viewing window so that all point(s) of intersection of the two graphs are visible in your viewing window.
ii. Copy the image in your viewing window onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Label the graphs with their equations. *Remember to draw all lines with a ruler.*
iii. Use the `intersect` utility to determine the coordinates of the point(s) of intersection. Plot the point of intersection on your homework paper and label it with its coordinates.
iv. Solve the equation $f(x)=k$ algebraically. Place your work and solution next to your graph. Do the solutions agree?

25. $f(x)=\sqrt{x-1}+\sqrt{x+6},\ k=7$

26. $f(x)=\sqrt{x+2}+\sqrt{x+9},\ k=7$

27. $f(x)=\sqrt{x+2}+\sqrt{3x+4},\ k=2$

28. $f(x)=\sqrt{6x+7}+\sqrt{3x+3},\ k=1$

In **Exercises 29-40**, use an algebraic technique to solve the given equation. Check your solutions.

29. $\sqrt{x+46}-\sqrt{x-35}=1$

30. $\sqrt{x-16}+\sqrt{x+16}=8$

31. $\sqrt{x-19}+\sqrt{x-6}=13$

32. $\sqrt{x+31}-\sqrt{x+12}=1$

33. $\sqrt{x-2}-\sqrt{x-49}=1$

34. $\sqrt{x+13}+\sqrt{x+8}=5$

35. $\sqrt{x+27}-\sqrt{x-22}=1$

36. $\sqrt{x+10}+\sqrt{x+13}=3$

37. $\sqrt{x+30}-\sqrt{x-38}=2$

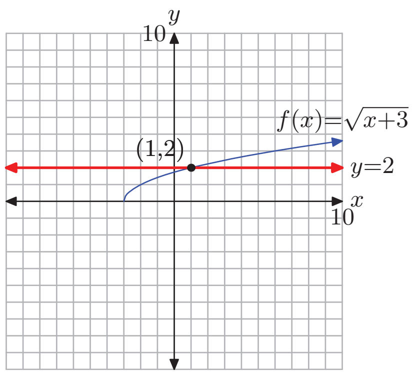
38. $\sqrt{x+36}-\sqrt{x+11}=1$

39. $\sqrt{x-17}+\sqrt{x+3}=10$

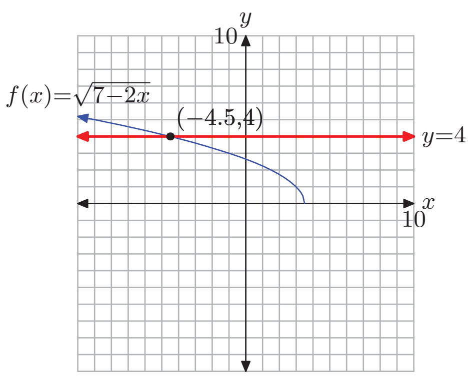
40. $\sqrt{x+18}+\sqrt{x+13}=5$

5.5 *Answers*

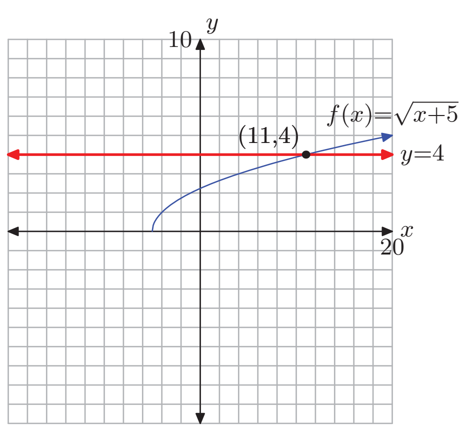
1. $x = 1$



3. $x = -9/2$



5. $x = 11$

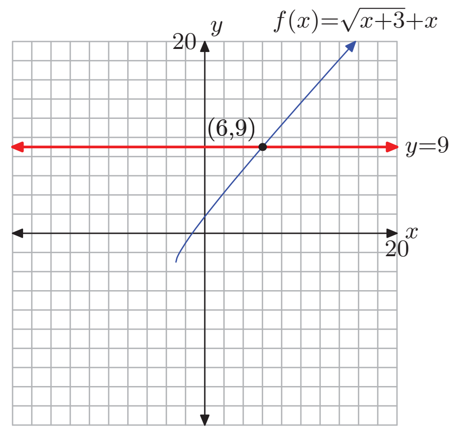


7. $\frac{1}{5}$

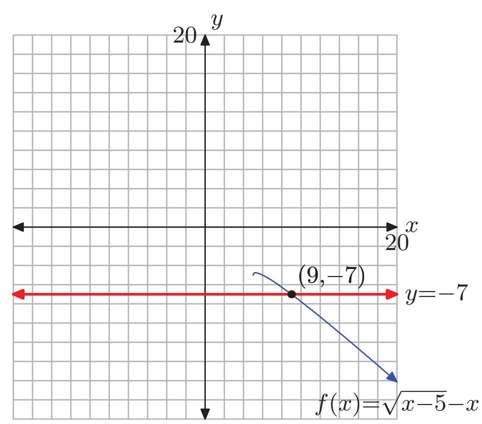
9. 12

11. $-\frac{8}{3}$

13. $x = 6$



15. $x = 9$



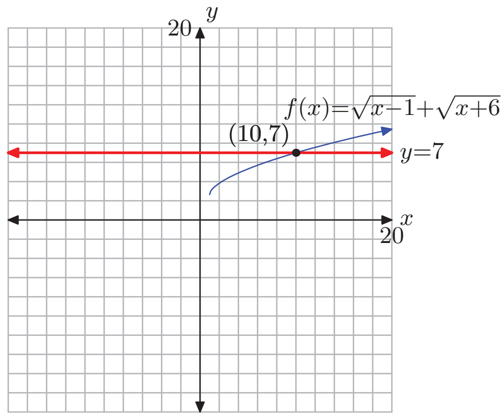
17. 3

19. 5

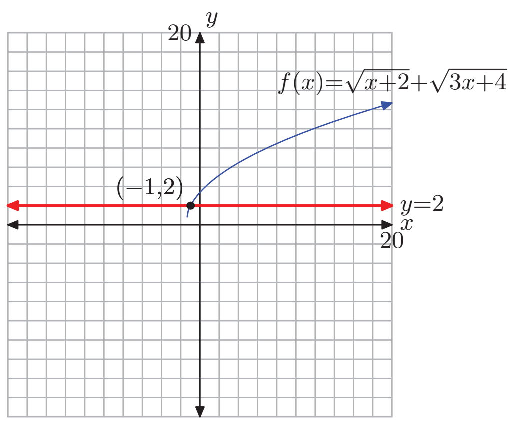
21. -1

23. $-8, -9$

25. $x = 10$



27. $x = -1$



29. 1635

31. 55

33. 578

35. 598

37. 294

39. 33

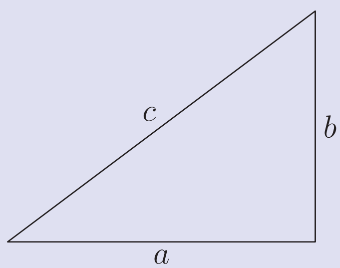
5.6 The Pythagorean Theorem

Pythagoras was a Greek mathematician and philosopher, born on the island of Samos (ca. 582 BC). He founded a number of schools, one in particular in a town in southern Italy called Crotone, whose members eventually became known as the Pythagoreans. The inner circle at the school, the *Mathematikoi*, lived at the school, rid themselves of all personal possessions, were vegetarians, and observed a strict vow of silence. They studied mathematics, philosophy, and music, and held the belief that numbers constitute the true nature of things, giving numbers a mystical or even spiritual quality.

Pythagoras.

Today, nothing is known of Pythagoras's writings, perhaps due to the secrecy and silence of the Pythagorean society. However, one of the most famous theorems in all of mathematics does bear his name, the *Pythagorean Theorem*.

Pythagorean Theorem. Let c represent the length of the **hypotenuse**, the side of a right triangle directly opposite the right angle (a right angle measures 90°) of the triangle. The remaining sides of the right triangle are called the **legs** of the right triangle, whose lengths are designated by the letters a and b.



The relationship involving the legs and hypotenuse of the right triangle, given by

$$a^2 + b^2 = c^2, \tag{1}$$

is called the **Pythagorean Theorem**.

Note that the Pythagorean Theorem can only be applied to right triangles.

Let's look at a simple application of the Pythagorean Theorem (1).

► **Example 2.** *Given that the length of one leg of a right triangle is 4 centimeters and the hypotenuse has length 8 centimeters, find the length of the second leg.*

Let's begin by sketching and labeling a right triangle with the given information. We will let x represent the length of the missing leg.

[15] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

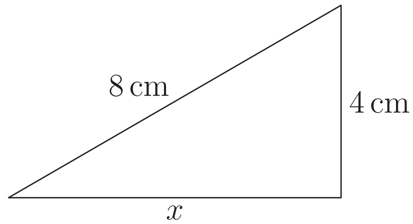


Figure 1. A sketch makes things a bit easier.

Here is an important piece of advice.

Tip 3. *The hypotenuse is the longest side of the right triangle. It is located directly opposite the right angle of the triangle. Most importantly, it is the quantity that is* **isolated** *by itself in the Pythagorean Theorem.*

$$a^2 + b^2 = c^2$$

Always isolate the quantity representing the hypotenuse on one side of the equation. The legs go on the other side of the equation.

So, taking the tip to heart, and noting the lengths of the legs and hypotenuse in **Figure 1**, we write

$$4^2 + x^2 = 8^2.$$

Square, then isolate x on one side of the equation.

$$\begin{aligned} 16 + x^2 &= 64 \\ x^2 &= 48 \end{aligned}$$

Normally, we would take plus or minus the square root in solving this equation, but x represents the length of a leg, which must be a positive number. Hence, we take just the positive square root of 48.

$$x = \sqrt{48}$$

Of course, place your answer in simple radical form.

$$\begin{aligned} x &= \sqrt{16}\sqrt{3} \\ x &= 4\sqrt{3} \end{aligned}$$

If need be, you can use your graphing calculator to approximate this length. To the nearest hundredth of a centimeter, $x \approx 6.93$ centimeters.

Proof of the Pythagorean Theorem

It is not known whether Pythagoras was the first to provide a proof of the Pythagorean Theorem. Many mathematical historians think not. Indeed, it is not even known if Pythagoras crafted a proof of the theorem that bears his name, let alone was the first to provide a proof.

There is evidence that the ancient Babylonians were aware of the Pythagorean Theorem over a 1000 years before the time of Pythagoras. A clay tablet, now referred to as Plimpton 322 (see **Figure 2**), contains examples of *Pythagorean Triples*, sets of three numbers that satisfy the Pythagorean Theorem (such as 3, 4, 5).

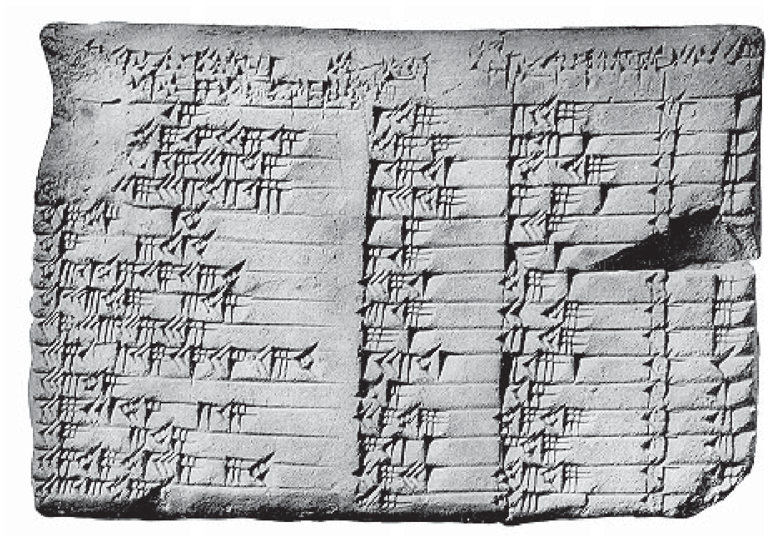


Figure 2. A photograph of Plimpton 322.

One of the earliest recorded proofs of the Pythagorean Theorem dates from the Han dynasty (206 BC to AD 220), and is recorded in the *Chou Pei Suan Ching* (see **Figure 3**). You can see that this figure specifically addresses the case of the 3, 4, 5 right triangle. Mathematical historians are divided as to whether or not the image was meant to be part of a general proof or was just devised to address this specific case. There is also disagreement over whether the proof was provided by a more modern commentator or dates back further in time.

However, **Figure 3** does suggest a path we might take on the road to a proof of the Pythagorean Theorem. Start with an arbitrary right triangle having legs of lengths a and b, and hypotenuse having length c, as shown in **Figure 4**(a).

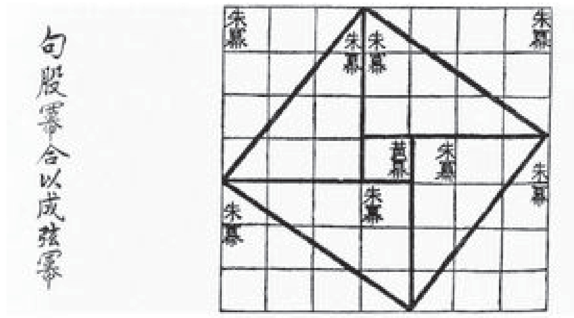


Figure 3. A figure from the Chou Pei Suan Ching.

Next, make four copies of the triangle shown in **Figure 4**(a), then rotate and translate them into place as shown in **Figure 4**(b). Note that this forms a big square that is c units on a side.

Further, the position of the triangles in **Figure 4**(b) allows for the formation of a smaller, unshaded square in the middle of the larger square. It is not hard to calculate the length of the side of this smaller square. Simply subtract the length of the smaller leg from the larger leg of the original triangle. Thus, the side of the smaller square has length $b - a$.

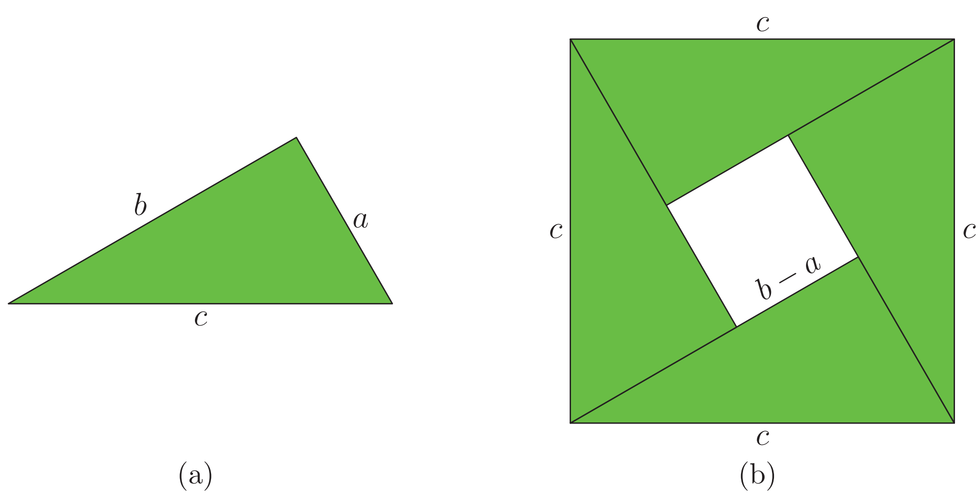


Figure 4. Proof of the Pythagorean Theorem.

Now, we will calculate the area of the large square in **Figure 4**(b) in two separate ways.

- First, the large square in **Figure 4**(b) has a side of length c. Therefore, the area of the large square is

$$\text{Area} = c^2.$$

- Secondly, the large square in **Figure 4**(b) is made up of 4 triangles of the same size and one smaller square having a side of length $b - a$. We can calculate the area of the large square by summing the area of the 4 triangles and the smaller square.

1. The area of the smaller square is $(b-a)^2$.
2. The area of each triangle is $ab/2$. Hence, the area of four triangles of equal size is four times this number; i.e., $4(ab/2)$.

   Thus, the area of the large square is

$$\begin{aligned}\text{Area} &= \text{Area of small square} + 4 \cdot \text{Area of triangle} \\ &= (b-a)^2 + 4\left(\frac{ab}{2}\right).\end{aligned}$$

We calculated the area of the larger square twice. The first time we got c^2; the second time we got $(b-a)^2 + 4(ab/2)$. Therefore, these two quantities must be equal.

$$c^2 = (b-a)^2 + 4\left(\frac{ab}{2}\right)$$

Expand the binomial and simplify.

$$\begin{aligned}c^2 &= b^2 - 2ab + a^2 + 2ab \\ c^2 &= b^2 + a^2\end{aligned}$$

That is,

$$a^2 + b^2 = c^2,$$

and the Pythagorean Theorem is proven.

Applications of the Pythagorean Theorem

In this section we will look at a few applications of the Pythagorean Theorem, one of the most applied theorems in all of mathematics. Just ask your local carpenter.

The ancient Egyptians would take a rope with 12 equally spaced knots like that shown in **Figure 5**, and use it to square corners of their buildings. The tool was instrumental in the construction of the pyramids.

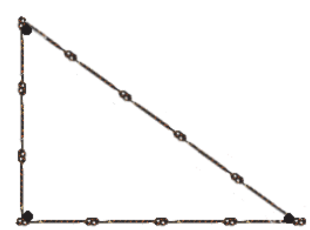


Figure 5. A basic 3-4-5 right triangle for squaring corners.

The Pythagorean theorem is also useful in surveying, cartography, and navigation, to name a few possibilities.

Let's look at a few examples of the Pythagorean Theorem in action.

► **Example 4.** *One leg of a right triangle is 7 meters longer than the other leg. The length of the hypotenuse is 13 meters. Find the lengths of all sides of the right triangle.*

Let x represent the length of one leg of the right triangle. Because the second leg is 7 meters longer than the first leg, the length of the second leg can be represented by the expression $x+7$, as shown in **Figure 6**, where we've also labeled the length of the hypotenuse (13 meters).

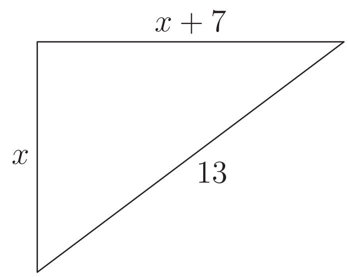


Figure 6. The second leg is 7 meters longer than the first.

Remember to isolate the length of the hypotenuse on one side of the equation representing the Pythagorean Theorem. That is,

$$x^2 + (x + 7)^2 = 13^2.$$

Note that the legs go on one side of the equation, the hypotenuse on the other. Square and simplify. Remember to use the squaring a binomial pattern.

$$\begin{aligned} x^2 + x^2 + 14x + 49 &= 169 \\ 2x^2 + 14x + 49 &= 169 \end{aligned}$$

This equation is nonlinear, so make one side zero by subtracting 169 from both sides of the equation.

$$\begin{aligned} 2x^2 + 14x + 49 - 169 &= 0 \\ 2x^2 + 14x - 120 &= 0 \end{aligned}$$

Note that each term on the left-hand side of the equation is divisible by 2. Divide both sides of the equation by 2.

$$x^2 + 7x - 60 = 0$$

Let's use the quadratic formula with $a = 1$, $b = 7$, and $c = -60$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-7 \pm \sqrt{7^2 - 4(1)(-60)}}{2(1)}$$

Simplify.

$$x = \frac{-7 \pm \sqrt{289}}{2}$$

Note that 289 is a perfect square ($17^2 = 289$). Thus,

$$x = \frac{-7 \pm 17}{2}.$$

Thus, we have two solutions,

$$x = 5 \quad \text{or} \quad x = -12.$$

Because length must be a positive number, we eliminate -12 from consideration. Thus, the length of the first leg is $x = 5$ meters. The length of the second leg is $x + 7$, or 12 meters.

Check. Checking is an easy matter. The legs are 5 and 12 meters, respectively, and the hypotenuse is 13 meters. Note that the second leg is 7 meters longer than the first. Also,

$$5^2 + 12^2 = 25 + 144 = 169,$$

which is the square of 13.

The integral sides of the triangle in the previous example, 5, 12, and 13, are an example of a *Pythagorean Triple*.

Pythagorean Triple. A set of positive integers a, b, and c, is called a Pythagorean Triple if they satisfy the Pythagorean Theorem; that is, if

$$a^2 + b^2 = c^2.$$

If the greatest common factor of a, b, and c is 1, then the triple (a, b, c) is called a **primitive** Pythagorean Triple.

Thus, for example, the Pythagorean Triple $(5, 12, 13)$ is primitive.

Let's look at another example.

► **Example 5.** *If (a, b, c) is a Pythagorean Triple, show that any positive integral multiple is also a Pythagorean Triple.*

Thus, if the positive integers (a, b, c) is a Pythagorean Triple, we must show that (ka, kb, kc), where k is a positive integer, is also a Pythagorean Triple.

However, we know that

$$a^2 + b^2 = c^2.$$

Multiply both sides of this equation by k^2.

$$k^2a^2 + k^2b^2 = k^2c^2$$

This last result can be written

$$(ka)^2 + (kb)^2 = (kc)^2.$$

Hence, (ka, kb, kc) is a Pythagorean Triple.

Hence, because $(3, 4, 5)$ is a Pythagorean Triple, you can double everything to get another triple $(6, 8, 10)$. Note that $6^2 + 8^2 = 10^2$ is easily checked. Similarly, tripling gives another triple $(9, 12, 15)$, and so on.

In **Example 5**, we showed that $(5, 12, 13)$ was a triple, so we can take multiples to generate other Pythagorean Triples, such as $(10, 24, 26)$ or $(15, 36, 39)$, and so on.

Formulae for generating Pythagorean Triples have been know since antiquity.

► **Example 6.** *The following formula for generating Pythagorean Triples was published in Euclid's (325–265 BC)* Elements*, one of the most successful textbooks in the history of mathematics. If m and n are positive integers with $m > n$, show*

$$\begin{aligned} a &= m^2 - n^2, \\ b &= 2mn, \\ c &= m^2 + n^2, \end{aligned} \tag{7}$$

generates Pythagorean Triples.

We need only show that the formulae for a, b, and c satisfy the Pythagorean Theorem. With that is mind, let's first compute $a^2 + b^2$.

$$\begin{aligned} a^2 + b^2 &= (m^2 - n^2)^2 + (2mn)^2 \\ &= m^4 - 2m^2n^2 + n^4 + 4m^2n^2 \\ &= m^4 + 2m^2n^2 + n^4 \end{aligned}$$

On the other hand,

$$\begin{aligned} c^2 &= (m^2 + n^2)^2 \\ &= m^4 + 2m^2n^2 + n^4. \end{aligned}$$

Hence, $a^2 + b^2 = c^2$, and the expressions for a, b, and c form a Pythagorean Triple.

It is both interesting and fun to generate Pythagorean Triples with the formulae from **Example 6**. Choose $m = 4$ and $n = 2$, then

$$\begin{aligned} a &= m^2 - n^2 = (4)^2 - (2)^2 = 12, \\ b &= 2mn = 2(4)(2) = 16, \\ c &= m^2 + n^2 = (4)^2 + (2)^2 = 20. \end{aligned}$$

It is easy to check that the triple $(12, 16, 20)$ will satisfy $12^2 + 16^2 = 20^2$. Indeed, note that this triple is a multiple of the basic $(3, 4, 5)$ triple, so it must also be a Pythagorean Triple.

It can also be shown that if m and n are relatively prime, and are not both odd or both even, then the formulae in **Example 6** will generate a **primitive** Pythagorean Triple. For example, choose $m = 5$ and $n = 2$. Note that the greatest common divisor of $m = 5$ and $n = 2$ is one, so m and n are relatively prime. Moreover, m is odd while n is even. These values of m and n generate

$$\begin{aligned} a &= m^2 - n^2 = (5)^2 - (2)^2 = 21, \\ b &= 2mn = 2(5)(2) = 20, \\ c &= m^2 + n^2 = (5)^2 + (2)^2 = 29. \end{aligned}$$

Note that

$$\begin{aligned} 21^2 + 20^2 &= 441 + 400 \\ &= 841 \\ &= 29^2. \end{aligned}$$

Hence, $(21, 20, 29)$ is a Pythagorean Triple. Moreover, the greatest common divisor of 21, 20, and 29 is one, so $(21, 20, 29)$ is primitive.

The practical applications of the Pythagorean Theorem are numerous.

▶ **Example 8.** *A painter leans a 20 foot ladder against the wall of a house. The base of the ladder is on level ground 5 feet from the wall of the house. How high up the wall of the house will the ladder reach?*

Consider the triangle in **Figure 7**. The hypotenuse of the triangle represents the ladder and has length 20 feet. The base of the triangle represents the distance of the base of the ladder from the wall of the house and is 5 feet in length. The vertical leg of the triangle is the distance the ladder reaches up the wall and the quantity we wish to determine.

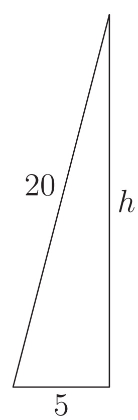


Figure 7. A ladder leans against the wall of a house.

Applying the Pythagorean Theorem,

$$5^2 + h^2 = 20^2.$$

Again, note that the square of the length of the hypotenuse is the quantity that is isolated on one side of the equation.

Next, square, then isolate the term containing h on one side of the equation by subtracting 25 from both sides of the resulting equation.

$$\begin{aligned} 25 + h^2 &= 400 \\ h^2 &= 375 \end{aligned}$$

We need only extract the positive square root.

$$h = \sqrt{375}$$

We could place the solution in simple form, that is, $h = 5\sqrt{15}$, but the nature of the problem warrants a decimal approximation. Using a calculator and rounding to the nearest tenth of a foot,

$$h \approx 19.4.$$

Thus, the ladder reaches about 19.4 feet up the wall.

The Distance Formula

We often need to calculate the distance between two points P and Q in the plane. Indeed, this is such a frequently recurring need, we'd like to develop a formula that will quickly calculate the distance between the given points P and Q. Such a formula is the goal of this last section.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two arbitrary points in the plane, as shown in **Figure 8**(a) and let d represent the distance between the two points.

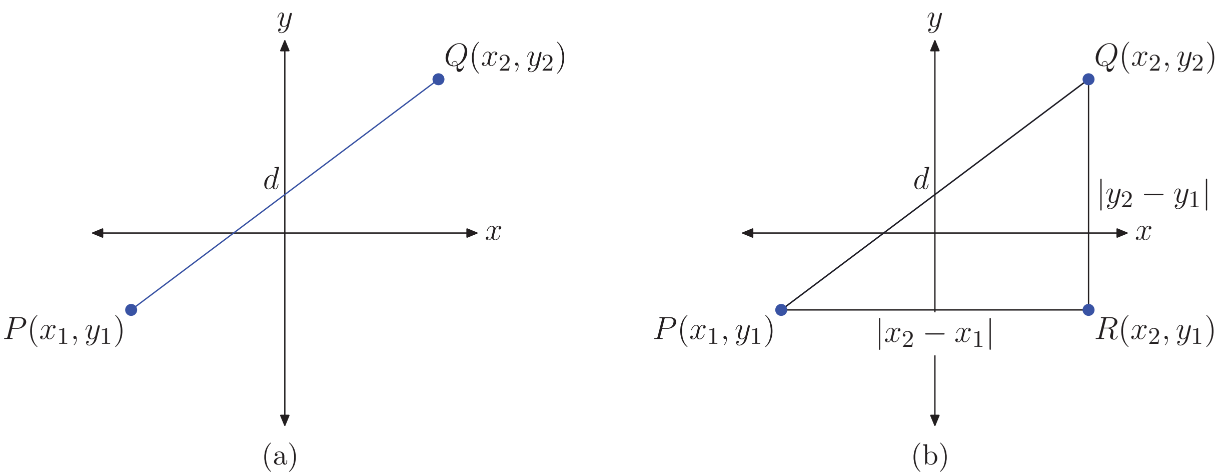


Figure 8. Finding the distance between the points P and Q.

To find the distance d, first draw the right triangle $\triangle PQR$, with legs parallel to the axes, as shown in **Figure 8**(b). Next, we need to find the lengths of the legs of the right triangle $\triangle PQR$.

- The distance between P and R is found by subtracting the x coordinate of P from the x-coordinate of R and taking the absolute value of the result. That is, the distance between P and R is $|x_2 - x_1|$.
- The distance between R and Q is found by subtracting the y-coordinate of R from the y-coordinate of Q and taking the absolute value of the result. That is, the distance between R and Q is $|y_2 - y_1|$.

We can now use the Pythagorean Theorem to calculate d. Thus,

$$d^2 = (|x_2 - x_1|)^2 + (|y_2 - y_1|)^2.$$

However, for any real number a,

$$(|a|)^2 = |a| \cdot |a| = |a^2| = a^2,$$

because a^2 is nonnegative. Hence, $(|x_2 - x_1|)^2 = (x_2 - x_1)^2$ and $(|y_2 - y_1|)^2 = (y_2 - y_1)^2$ and we can write

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Taking the positive square root leads to the *Distance Formula*.

The Distance Formula. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two arbitrary points in the plane. The distance d between the points P and Q is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{9}$$

The direction of subtraction is unimportant. Because you square the result of the subtraction, you get the same response regardless of the direction of subtraction (e.g. $(5 - 2)^2 = (2 - 5)^2$). Thus, it doesn't matter which point you designate as the point P, nor does it matter which point you designate as the point Q. Simply subtract x-coordinates and square, subtract y-coordinates and square, add, then take the square root.

Let's look at an example.

► **Example 10.** *Find the distance between the points $P(-4, -2)$ and $Q(4, 4)$.*

It helps the intuition if we draw a picture, as we have in **Figure 9**. One can now take a compass and open it to the distance between points P and Q. Then you can place your compass on the horizontal axis (or any horizontal gridline) to estimate the distance between the points P and Q. We did that on our graph paper and estimate the distance $d \approx 10$.

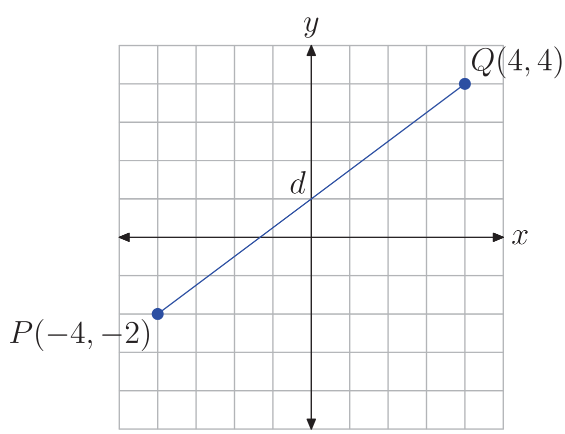


Figure 9. Gauging the distance between $P(-4, -2)$ and $Q(4, 4)$.

Let's now use the distance formula to obtain an exact value for the distance d. With $(x_1, y_1) = P(-4, -2)$ and $(x_2, y_2) = Q(4, 4)$,

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(4 - (-4))^2 + (4 - (-2))^2} \\ &= \sqrt{8^2 + 6^2} \\ &= \sqrt{64 + 36} \\ &= \sqrt{100} \\ &= 10. \end{aligned}$$

It's not often that your exact result agrees with your approximation, so never worry if you're off by just a little bit.

5.6 *Exercises*

In **Exercises 1-8**, state whether or not the given triple is a Pythagorean Triple. Give a reason for your answer.

1. $(8, 15, 17)$

2. $(7, 24, 25)$

3. $(8, 9, 17)$

4. $(4, 9, 13)$

5. $(12, 35, 37)$

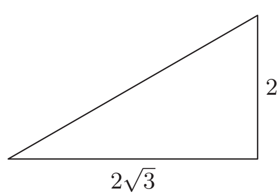
6. $(12, 17, 29)$

7. $(11, 17, 28)$

8. $(11, 60, 61)$

In **Exercises 9-16**, set up an equation to model the problem constraints and solve. Use your answer to find the missing side of the given right triangle. Include a sketch with your solution and check your result.

9.



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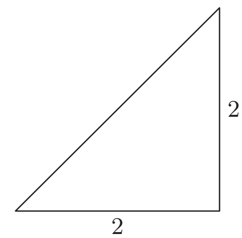

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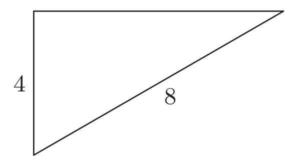

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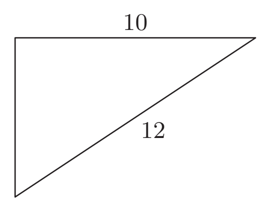

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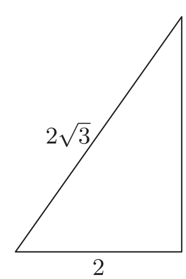

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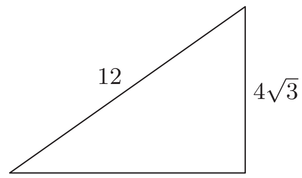

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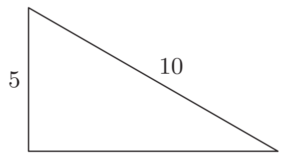
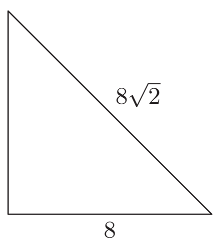


16.



In **Exercises 17-20**, set up an equation that models the problem constraints. Solve the equation and use the result to answer the question. Look back and check your result.

17. The legs of a right triangle are consecutive positive integers. The hypotenuse has length 5. What are the lengths of the legs?

18. The legs of a right triangle are consecutive even integers. The hypotenuse has length 10. What are the lengths of the legs?

19. One leg of a right triangle is 1 centimeter less than twice the length of the first leg. If the length of the hypotenuse is 17 centimeters, find the lengths of the legs.

20. One leg of a right triangle is 3 feet longer than 3 times the length of the first leg. The length of the hypotenuse is 25 feet. Find the lengths of the legs.

21. Pythagoras is credited with the following formulae that can be used to generate Pythagorean Triples.

$$a = m$$
$$b = \frac{m^2 - 1}{2},$$
$$c = \frac{m^2 + 1}{2}$$

Use the technique of Example 6 to demonstrate that the formulae given above will generate Pythagorean Triples, provided that m is an *odd* positive integer larger than one. Secondly, generate at least 3 instances of Pythagorean Triples with Pythagoras's formula.

22. Plato (380 BC) is credited with the following formulae that can be used to generate Pythagorean Triples.

$$a = 2m$$
$$b = m^2 - 1,$$
$$c = m^2 + 1$$

Use the technique of Example 6 to demonstrate that the formulae given above will generate Pythagorean Triples, provided that m is a positive integer larger than 1. Secondly, generate at least 3 instances of Pythagorean Triples with Plato's formula.

In **Exercises 23-28**, set up an equation that models the problem constraints. Solve the equation and use the result to answer the question. Look back and check your result.

23. Fritz and Greta are planting a 12-foot by 18-foot rectangular garden, and are laying it out using string. They would like to know the length of a diagonal to make sure that right angles are formed. Find the length of a diagonal. Approximate your answer to within 0.1 feet.

24. Angelina and Markos are planting a 20-foot by 28-foot rectangular garden, and are laying it out using string. They would like to know the length of a diagonal to make sure that right angles are formed. Find the length of a diagonal. Approximate your answer to within 0.1 feet.

25. The base of a 36-foot long guy wire is located 16 feet from the base of the telephone pole that it is anchoring. How high up the pole does the guy wire reach? Approximate your answer to within 0.1 feet.

26. The base of a 35-foot long guy wire is located 10 feet from the base of the telephone pole that it is anchoring. How high up the pole does the guy wire reach? Approximate your answer to within 0.1 feet.

27. A stereo receiver is in a corner of a 13-foot by 16-foot rectangular room. Speaker wire will run under a rug, diagonally, to a speaker in the far corner. If 3 feet of slack is required on each end, how long a piece of wire should be purchased? Approximate your answer to within 0.1 feet.

28. A stereo receiver is in a corner of a 10-foot by 15-foot rectangular room. Speaker wire will run under a rug, diagonally, to a speaker in the far corner. If 4 feet of slack is required on each end, how long a piece of wire should be purchased? Approximate your answer to within 0.1 feet.

In **Exercises 29-38**, use the distance formula to find the exact distance between the given points.

29. $(-8, -9)$ and $(6, -6)$

30. $(1, 0)$ and $(-9, -2)$

31. $(-9, 1)$ and $(-8, 7)$

32. $(0, 9)$ and $(3, 1)$

33. $(6, -5)$ and $(-9, -2)$

34. $(-9, 6)$ and $(1, 4)$

35. $(-7, 7)$ and $(-3, 6)$

36. $(-7, -6)$ and $(-2, -4)$

37. $(4, -3)$ and $(-9, 6)$

38. $(-7, -1)$ and $(4, -5)$

In **Exercises 39-42**, set up an equation that models the problem constraints. Solve the equation and use the result to answer the question. Look back and check your result.

39. Find k so that the point $(4, k)$ is $2\sqrt{2}$ units away from the point $(2, 1)$.

40. Find k so hat the point $(k, 1)$ is $2\sqrt{2}$ units away from the point $(0, -1)$.

41. Find k so that the point $(k, 1)$ is $\sqrt{17}$ units away from the point $(2, -3)$.

42. Find k so that the point $(-1, k)$ is $\sqrt{13}$ units away from the point $(-4, -3)$.

---

43. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Plot the points $P(0, 5)$ and $Q(4, -3)$ on your coordinate system.

a) Plot several points that are equidistant from the points P and Q on your coordinate system. What graph do you get if you plot **all** points that are equidistant from the points P and Q? Determine the equation of the graph by examining the resulting image on your coordinate system.

b) Use the distance formula to find the equation of the graph of all points that are equidistant from the points P and Q. *Hint: Let (x, y) represent an arbitrary point on the graph of all points equidistant from points P and Q. Calculate the distances from the point (x, y) to the points P and Q separately, then set them equal and simplify the resulting equation. Note that this analytical approach should provide an equation that matches that found by the graphical approach in part (a).*

44. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Plot the point $P(0, 2)$ and label it with its coordinates. Draw the line $y = -2$ and label it with its equation.

a) Plot several points that are equidistant from the point P and the line $y = -2$ on your coordinate system. What graph do you get if you plot **all** points that are equidistant from the points P and the line $y = -2$.

b) Use the distance formula to find the equation of the graph of all points that are equidistant from the points P and the line $y = -2$. *Hint: Let (x, y) represent an arbitrary point on the graph of all points equidistant from points P and the line $y = -2$. Calculate the distances from the point (x, y) to the points P and the line $y = -2$ separately, then set them equal and simplify the resulting equation.*

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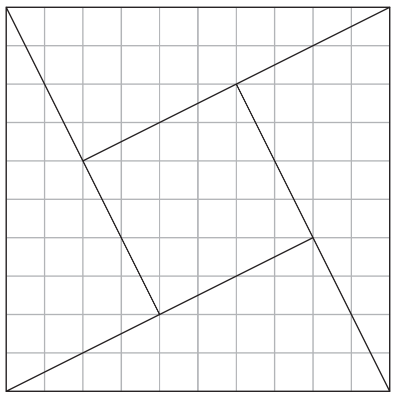
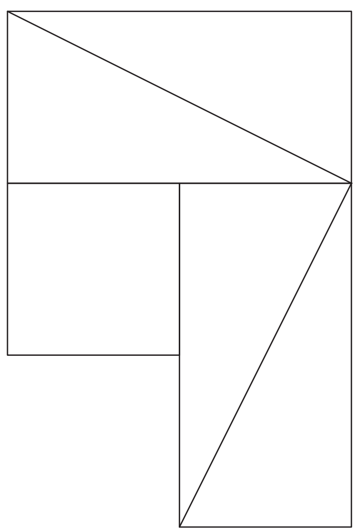
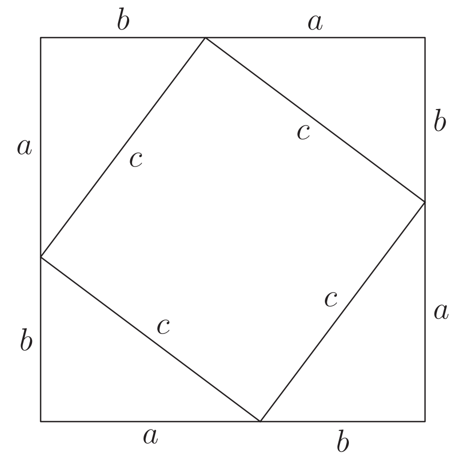
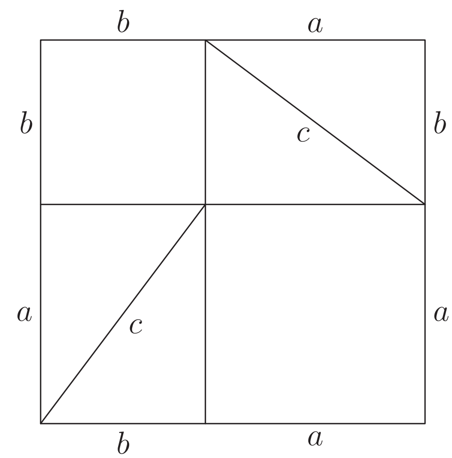
45. Copy the following figure onto a sheet of graph paper. Cut the pieces of the first figure out with a pair of scissors, then rearrange them to form the second figure. Explain how this proves the Pythagorean Theorem.





46. Compare this image to the one that follows and explain how this proves the Pythagorean Theorem.





5.6 *Answers*

1. Yes, because $8^2 + 15^2 = 17^2$

3. No, because $8^2 + 9^2 \neq 17^2$

5. Yes, because $12^2 + 35^2 = 37^2$

7. No, because $11^2 + 17^2 \neq 28^2$

9. 4

11. $4\sqrt{3}$

13. $2\sqrt{2}$

15. $5\sqrt{3}$

17. The legs have lengths 3 and 4.

19. The legs have lengths 8 and 15 centimeters.

21. $(3, 4, 5)$, $(5, 12, 13)$, and $(7, 24, 25)$, with $m = 3$, 5, and 7, respectively.

23. 21.63 ft

25. 32.25 ft

27. 26.62 ft

29. $\sqrt{205}$

31. $\sqrt{37}$

33. $\sqrt{234} = 3\sqrt{26}$

35. $\sqrt{17}$

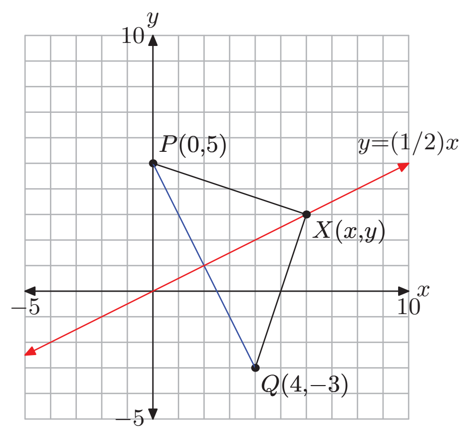
37. $\sqrt{250} = 5\sqrt{10}$

39. $k = 3, -1$.

41. $k = 1, 3$.

43.

a) In the figure that follows, $XP = XQ$.



b) $y = (1/2)x$